Solutions to Week 6 Homework

ASSIGNMENT 13.

3.1.19. *Find the solution of the initial value problem*

$$y'' - y = 0, \quad y(0) = \frac{5}{4}, \; y'(0) = -\frac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.[5 points for the solution, 2 for the plot, 3 for the minimum value.]

The characteristic equation is

$$r^2 - 1 = 0,$$

which has roots $r = \pm 1$. Thus, a fundamental set of solutions is

$$y_1 = e^t, \quad y_2 = e^{-t}.$$

We look for a solution of the form

$$y = C_1 e^t + C_2 e^{-t}.$$

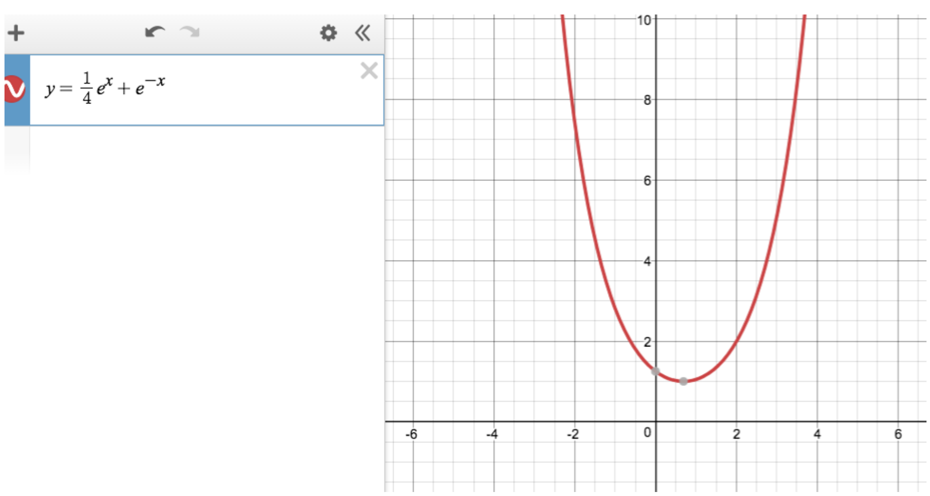
To solve the required initial value problem, we need

$$\frac{5}{4} = C_1 + C_2,$$
$$-\frac{3}{4} = C_1 - C_2.$$

Adding these equations together gives $1/2 = 2C_1$, or $C_1 = 1/4$, and subtracting them gives $2 = 2C_2$, or $C_2 = 1$. So the solution is

$$y = \frac{1}{4}e^t + e^{-t}.$$

Here is a graph:



As you can see, the function has a unique critical point, which is a minimum. This should be the unique point where $y' = 0$. We have

$$y' = \frac{1}{4}e^t - e^{-t},$$

so we need

$$\begin{aligned}\frac{1}{4}e^t &= e^{-t}\\ e^{-2t} &= 1/4\\ -2t &= \ln(1/4)\\ t &= \ln(2) \approx 0.693.\end{aligned}$$

[It is fine to read this off the graph rather than solving, though you might want to point out that they can solve it exactly.]

22. *Solve the initial value problem*

$$4y'' - y = 0, \quad y(0) = 2,\ y'(0) = \beta.$$

Then find β so that the solution approaches zero as $t \to \infty$. [5 points for the solution, 5 for $\beta = -1$.]

The characteristic equation is $4r^2 - 1 = 0$, which has solutions $r = \pm 1/2$. So the general solution is

$$y = C_1 e^{t/2} + C_2 e^{-t/2}.$$

To solve the initial value problem, we solve the system of equations

$$\begin{aligned}2 &= C_1 + C_2,\\ \beta &= \frac{C_1}{2} - \frac{C_2}{2}.\end{aligned}$$

The second equation gives

$$2\beta = C_1 - C_2.$$

Adding this to the first equation shows that $C_1 = \beta + 1$. Subtracting the two equations shows that $C_2 = 1 - \beta$. So the solution is

$$y = (1+\beta)e^{t/2} + (1-\beta)e^{-t/2}.$$

As $t \to \infty$, the first term will predominate, and send y to $\pm\infty$, unless its coefficient is zero. So we need $\beta = -1$, which makes

$$y = 2e^{-t/2}.$$

This does, in fact, go to 0 as $t \to \infty$.

ASSIGNMENT 14.

3.2.13. *Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation*

$$t^2 y'' - 2y = 0$$

for $t > 0$. Then show that $y = c_1 t^2 + c_2 t^{-1}$ is also a solution of this equation for any c_1 and c_2.[5 pts – okay to quote the theorem for the second part]

We have

$$t^2 y_1'' - 2y_1 = t^2 \cdot 2 - 2t^2 = 0$$

and

$$t^2 y_2'' - 2y_2 = t^2 \cdot (2t^{-3}) - 2t^{-1} = 0.$$

So both are solutions for $t > 0$. (At $t = 0$, y_2 has a discontinuity. However, y_2 is also a solution for $t < 0$, and y_1 is a solution that is valid for all t.)

Since this is a linear and homogeneous equation, its set of solutions is closed under linear combinations, by Theorem 3.2.2 in the text. Thus, any $c_1y_1 + c_2y_2$ is also a solution.

Alternatively, we could check directly that $y = c_1y_1 + c_2y_2$ is a solution. We have

$$\begin{aligned} y &= c_1t^2 + c_2t^{-1} \\ y' &= 2c_1t - c_2t^{-2} \\ y'' &= 2c_1 + 2c_2t^{-3} \end{aligned}$$

and thus

$$t^2y'' - 2y = 2c_1t^2 + 2c_2t^{-1} - 2c_1t^2 - 2c_2t^{-1} = 0.$$

14. *Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation*

$$yy'' + (y')^2 = 0$$

for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution to the equation. Explain why this does not contradict Theorem 3.2.2.[7 pts]

We check that

$$y_1y_1'' + (y_1')^2 = 1 \cdot 0 + 0^2 = 0$$

and that

$$y_2y_2'' + (y_2')^2 = t^{1/2} \cdot (1/2)(-1/2)t^{-3/2} + \left[(1/2)t^{-1/2}\right]^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0.$$

For a general $y = c_1 + c_2t^{1/2}$, we have

$$\begin{aligned} y' &= \frac{c_2}{2}t^{-1/2} \\ y'' &= -\frac{c_2}{4}t^{-3/2} \end{aligned}$$

Therefore,

$$yy'' + (y')^2 = -\frac{c_1c_2}{4}t^{-3/2} - \frac{c_2^2}{4}t^{-1} + \frac{c_2^2}{4}t^{-1} = -\frac{c_1c_2}{4}t^{-3/2}.$$

This is only zero if c_1 or c_2 is zero. Another way to approach this would be to pick a random value for c_1 and c_2 and try it out – typically with problems like this, it's very unlikely that you would land on the wrong answer by chance.

This does not contradict Theorem 3.2.2 because the differential equation is not linear. It is usually only true for linear, homogeneous differential equations that their solutions are closed under taking linear combinations.

25. *Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?* [8 pts – 4 for verification, 4 for checking that they are a fundamental set of solutions.]

$$y'' - 2y' + y = 0; \quad y_1(t) = e^t,\ y_2(t) = te^t.$$

We check that

$$y_1'' - 2y_1' + y_1 = e^t - 2e^t + e^t = 0.$$

For y_2, we have $y_2' = e^t + te^t$ by the product rule, and $y_2'' = 2e^t + te^t$ by the same method. So

$$y_2'' - 2y_2' + y_2 = (2e^t + te^t) - 2(e^t + te^t) + te^t = 0.$$

Absent a method of solving this equation in general, we can prove that these are a fundamental set of solutions by calculating the Wronskian. We have

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}.$$

This is never zero, so y_1 and y_2 are a fundamental set of solutions.

Note that Abel's theorem tells us that the Wronskian of y_1 and y_2 is of the form Ce^{2t} just by looking at the equation. However, it does not tell us the value of C, and in particular whether or not $C = 0$. So Abel's theorem by itself is of no help here.

Another way to do this is to notice that neither of y_1 or y_2 is a scalar multiple of each other, so they are linearly independent. Since the solutions to a linear homogeneous second-order equation are always a two-dimensional vector space – they are always generated by a fundamental set of two solutions – it follows immediately that y_1 and y_2 are a fundamental set of solutions. (If not, we could find a third linearly independent solution y_3, which would give us the general solution in terms of three parameters. But this doesn't happen for linear homogeneous second-order equations.)