Week 7 solutions

ASSIGNMENT 15.

3.3.17. *Find the solution of the initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.*

$$y'' + 4y = 0, \quad y(0) = 0,\ y'(0) = 1.$$

The characteristic equation is

$$r^2 + 4 = 0,$$

which has complex conjugate roots,

$$r = \pm 2i.$$

This means that a general complex-valued solution to the problem is given by

$$y = C_1 e^{2it} + C_2 e^{-2it}, \quad C_1, C_2 \in \mathbb{C}.$$

As usual, we can find a fundamental set of real-valued solutions by taking the real and imaginary part of *either* of the complex-valued solutions in the fundamental set $\{e^{2it}, e^{-2it}\}$. For example, we can define

$$y_1 = \operatorname{Re}(e^{2it}) = \cos(2t),$$
$$y_2 = \operatorname{Im}(e^{2it}) = \sin(2t).$$

As a reminder, $\operatorname{Im}(e^{2it})$ is $\sin(2t)$ and not $i\sin(2t)$ (so that it is real-valued, not imaginary-valued); also, if we performed the same operations on e^{-2it}, we'd get scalar multiples of these two solutions.

So the general solution is

$$y = C_1 \cos(2t) + C_2 \sin(2t).$$

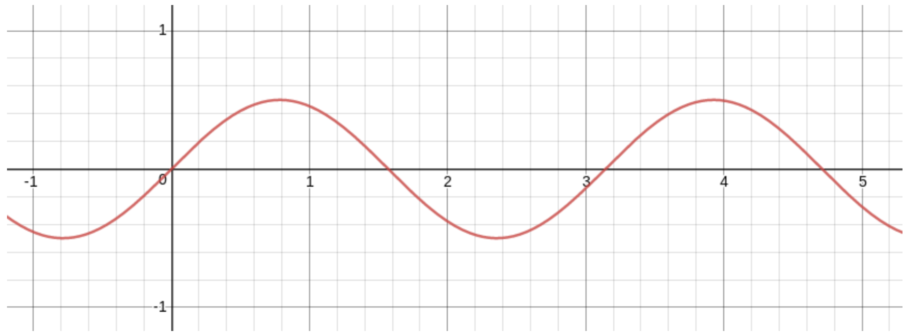
For this y, we have

$$y(0) = C_1, \quad y'(0) = 2C_2.$$

So to solve the initial value problem, we put $C_1 = 0$ and $C_2 = 1/2$. Then

$$y = \frac{1}{2}\sin(2t).$$

This solution oscillates forever as t increases, with amplitude $1/2$ and period π.



23. *Consider the initial value problem*

$$3u'' - u' + 2u = 0, \quad u(0) = 2, u'(0) = 0.$$

(a) *Find the solution $u(t)$ of this problem.*
The characteristic equation is

$$3r^2 - r + 2 = 0.$$

Using the quadratic formula, we obtain

$$r = \frac{1 \pm \sqrt{1-24}}{6} = \frac{1 \pm i\sqrt{23}}{6}.$$

This means that a general real-valued solution is given by

$$u = C_1 e^{t/6} \cos\left(\frac{\sqrt{23}}{6}t\right) + C_2 e^{t/6} \sin\left(\frac{\sqrt{23}}{6}t\right).$$

We have

$$u(0) = C_1, \quad u'(0) = \frac{C_1}{6} + C_2\frac{\sqrt{23}}{6}.$$

Solving with the given initial values, we get

$$C_1 = 2, \quad C_2 = \frac{-2}{\sqrt{23}}.$$

So the answer is

$$u = 2e^{t/6} \cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{\sqrt{23}} e^{t/6} \sin\left(\frac{\sqrt{23}}{6}t\right).$$

(b) *For $t > 0$, find the first time at which $|u(t)| = 10$.*
This should probably be done with Desmos or other computer software. I got $t \approx 14.106$.

ASSIGNMENT 16.

3.4.15. *Consider the initial value problem*

$$4y'' + 12y' + 9y = 0, \quad y(0) = 1, \; y'(0) = -4.$$

(a) *Solve the initial value problem and plot its solution for* $0 \le t \le 5$.

The characteristic polynomial is

$$4r^2 + 12r + 9 = 0,$$

which factors as

$$(2r+3)^2 = 0.$$

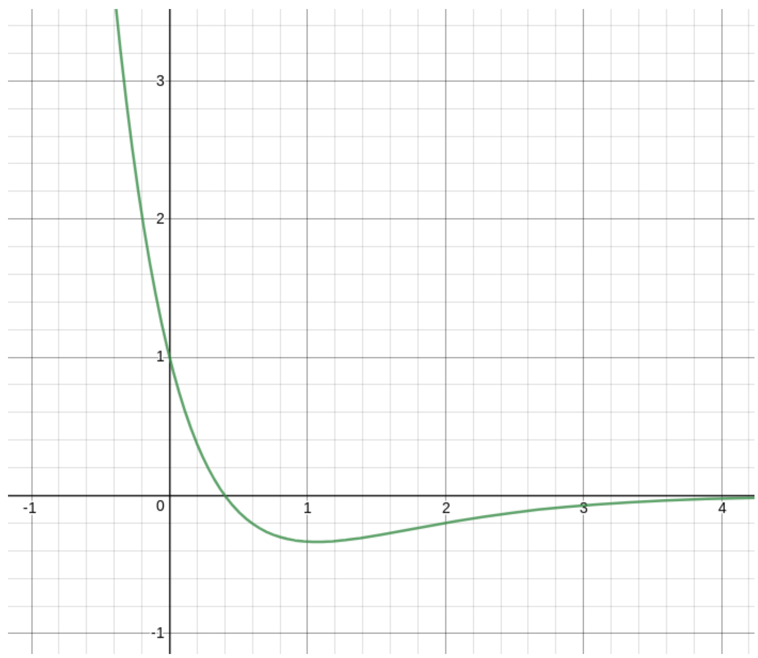
This has the repeated root $r = -3/2$. So the general solution is

$$y = C_1 e^{-3t/2} + C_2 t e^{-3t/2}.$$

We have $1 = y(0) = C_1$, and $-4 = y'(0) = -3C_1/2 + C_2$. Thus, the solution to the initial value problem is

$$y = e^{-3t/2} - \frac{5}{2} t e^{-3t/2}.$$

Here is a graph:



It appears that the solution crosses the t-axis once and then approaches zero asymptotically from the bottom.

(b) *Determine where the solution takes the value zero.*

We can solve the equation explicitly:

$$0 = e^{-3t/2} - \frac{5}{2} t e^{-3t/2},$$

and dividing by $e^{-3t/2}$, which is never zero, we get

$$0 = 1 - 5t/2$$

or $t = 2/5$. This is the only zero.

(c) *Determine the coordinates (t_0, y_0) of the minimum point.*

We have

$$y' = -4e^{-3t/2} + \frac{15}{4}te^{-3t/2}.$$

This is zero when $0 = -4 + 15t/4$, or $t = 16/15$. Since this is the only critical point of the function, it must be the minimum shown on the graph above.

(d) *Change the second condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.*

If $y'(0) = b$ and $y(0) = 1$, then the equation for y is

$$y = e^{-3t/2} + (b + 3/2)te^{-3t/2}.$$

This has a zero where $0 = 1 + (b + 3/2)t$, or $t = -1/(b + 3/2)$. For $b < -3/2$, this value of t is positive, and for $b > -3/2$, this value of t is negative. So for $b > -3/2$, y never crosses the t-axis at a positive value of t, meaning that it is always positive for $t > 0$. (The problem is a little imprecise in saying that the function "always remains positive", but this is the only reasonable way to interpret it.) So the critical value of b is $b = -3/2$. Note that for $b = -3/2$, the solution is never zero for any value of t, positive or negative.

25. *Use the method of reduction of order to find a second solution of the differential equation.*

$$t^2y'' + 3ty' + y = 0, \quad t > 0; \ y_1(t) = t^{-1}.$$

First check that y_1 is actually a solution of the equation:

$$t^2y_1'' + 3ty_1' + y_1 = 2t^2 \cdot t^{-3} - 3t \cdot t^{-2} + t^{-1} = 0.$$

Now define $y_2 = vy_1 = t^{-1}v$. Then

$$\begin{aligned} y_2' &= -t^{-2}v + t^{-1}v', \\ y_2'' &= 2t^{-3}v - 2t^{-2}v' + t^{-1}v''. \end{aligned}$$

Substituting into the equation gives

$$0 = 2t^{-1}v - 2v' + tv'' - 3t^{-1}v + 3v' + t^{-1}v = v' + tv''.$$

As expected, the terms involving v cancel out. Let $w = v'$; then the equation is first-order in terms of w and separable, and we can solve it:

$$\begin{aligned} 0 &= w + tw' \\ -w &= tw' \\ \int -\frac{1}{w}\,dw &= \int \frac{1}{t}\,dt \\ -\ln|w| &= \ln(t) + C \quad (t > 0) \\ |w| &= At^{-1} \quad (A > 0) \\ w &= At^{-1} \quad (A \text{ arbitrary}) \end{aligned}$$

Since $w = v'$, we integrate this again to get $v = A\ln(t) + B$. One such solution is $v = \ln(t)$. Then $y_2 = vy_1 = t^{-1}\ln(t)$. (Note that the general solution to the equation is obtained by keeping the general solution for v: $y = At^{-1}\ln(t) + Bt^{-1}$.)

ASSIGNMENT 17.

3.5.15. *Find the solution of the given initial value problem.*

$$y'' + y' - 2y = 2t, \quad y(0) = 0,\ y'(0) = 1.$$

First, we solve the associated homogeneous equation,

$$y'' + y' - 2y = 0.$$

The characteristic polynomial, $r^2 + r - 2$, has roots $r = -2$ and $r = 1$. So the general solution is

$$y = C_1 e^{-2t} + C_2 e^t.$$

Now, apply the method of undetermined coefficients to find a particular solution to the inhomogeneous equation. Since the right-hand side is a polynomial in t of degree 1, we should try substituting this for y. So let $y = At + B$. Then $y' = A$ and $y'' = 0$. We get

$$y'' + y' - 2y = A - 2At - 2B = 2t.$$

Comparing coefficients, we see that $A = -1$ and $B = -1/2$. This gives the particular solution

$$y_p = -t - 1/2.$$

Thus, the general solution to the inhomogeneous equation is

$$y = -t - 1/2 + C_1 e^{-2t} + C_2 e^t.$$

We have $y(0) = -1/2 + C_1 + C_2 = 0$, and $y'(0) = -1 - 2C_1 + C_2 = 1$. So $C_1 = -1/2$ and $C_2 = 1$. So, the solution is

$$y = -t - 1/2 - \frac{1}{2} e^{-2t} + e^t.$$

21(a) *Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.*

$$y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin(3t).$$

First, solve the associated homogeneous equation. This is $y'' + 3y' = 0$, and its solutions are $y = C_1 e^{-3t} + C_2$.

Next, let y_1, y_2, and y_3 be solutions to the three inhomogeneous equations

$$\begin{aligned} y_1'' + 3y_1' &= 2t^4, \\ y_2'' + 3y_2' &= t^2 e^{-3t}, \\ y_3'' + 3y_3' &= \sin(3t). \end{aligned}$$

Then $y_1 + y_2 + y_3$ is a solution to the original equation. So it suffices to find the right form for the three separate equations above.

The right-hand side of the equation solved by y_1 is a polynomial of degree 4. So we should try a general polynomial of degree 4, $y_1 = A_0 t^4 + A_1 t^3 + A_2 t^2 + \cdots + A_4$. However, the last term, A_4, is a solution to the associated homogeneous equation, so we will not be able to solve for all the coefficients of y_1 this way. (Try it if you don't believe me.) To avoid the solutions to the associated homogeneous equation, we multiply by t:

$$y_1 = A_0 t^5 + A_1 t^4 + \cdots + A_4 t.$$

For y_2, we ought to try a polynomial of degree 2 times e^{-3t}. Again, we will need to multiply by t to avoid the solution space to the associated homogeneous equation. We ought to use

$$y_2 = B_0 t^3 e^{-3t} + B_1 t^2 e^{-3t} + B_2 t e^{-3t}.$$

Finally, for y_3, we should use a general linear combination of $\sin(3t)$ and $\cos(3t)$.

$$y_3 = C_0 \sin(3t) + C_1 \cos(3t).$$

So the appropriate form for y is

$$y = A_0 t^5 + A_1 t^4 + \cdots + A_4 t + B_0 t^3 e^{-3t} + B_1 t^2 e^{-3t} + B_2 t e^{-3t} +_0 \sin(3t) + C_1 \cos(3t).$$

22(a) *Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.*

$$y'' + y = t(1 + \sin t).$$

The solutions to the associated homogeneous equation are $y = C_1 \sin(t) + C_2 \cos(t)$.

Like the previous problem, we should break this one into parts, one where the right-hand side is t and one where it is $t \sin(t)$. If

$$y_1'' + y_1 = t,$$

then y_1 will have the form

$$y_1 = A_0 t + A_1,$$

a general linear polynomial in t. If

$$y_2'' + y_2 = t \sin(t),$$

then we want to try a general linear polynomial in t times a general linear combination of $\sin(t)$ and $\cos(t)$, i.e.,

$$y_2 = B_0 t \sin(t) + B_1 \sin(t) + B_2 t^2 \cos(t) + B_3 \cos(t).$$

However, some of these terms overlap with the solution to the associated homogeneous equation. So we must multiply by t, giving

$$y_2 = B_0 t^2 \sin(t) + B_1 t \sin(t) + B_2 t^2 \cos(t) + B_3 t \cos(t).$$

Therefore,

$$y = A_0 t + A_1 + B_0 t^2 \sin(t) + B_1 t \sin(t) + B_2 t^2 \cos(t) + B_1 t \cos(t).$$

The part involving the B's is a general trig function times a general polynomial of degree 1.