Week 8 solutions

ASSIGNMENT 18.

3.6.15. *Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation. Then find a particular solution of the given nonhomogeneous equation.*

$$ty'' - (1+t)y' + y = t^2e^{2t},\ t > 0; \quad y_1(t) = 1+t,\ y_2(t) = e^t.$$

We have

$$ty_1'' - (1+t)y_1' + y_1 = t \cdot 0 - (1+t) \cdot 1 + (1+t) = 0,$$

and

$$ty_2'' - (1+t)y_2' + y_2 = te^t - (1+t)e^t + e^t = 0.$$

So the general solution to the homogeneous equation is

$$y = C_1(1+t) + C_2e^t.$$

Now define

$$y = u_1 \cdot (1+t) + u_2e^t,$$

where u_1 and u_2 are unknown functions. We will impose the relation

$$u_1' \cdot (1+t) + u_2'e^t = 0.$$

Then

$$y' = u_1 + u_2e^t,$$

and

$$y'' = u_1' + u_2'e^t + u_2e^t.$$

Substituting these into the inhomogeneous equation gives

$$tu_1' + tu_2'e^t + tu_2e^t - (1+t)(u_1 + u_2e^t) + (1+t)u_1 + u_2e^t = t^2e^{2t},$$

or

$$u_1' + u_2'e^t = te^{2t},$$

where all the terms involving u_1 and u_2 cancel. So $u_1' = te^{2t} - u_2'e^t$. Substituting this into the assumed relation

$$(1+t)u_1' + u_2'e^t = 0$$

gives

$$(1+t)te^{2t} - te^tu_2' = 0$$

or

$$u_2' = (1+t)e^t.$$

This means that $u_1' = -e^{2t}$. Integrating, we get

$$u_1 = -\frac{1}{2}e^{2t} \text{ and } u_2 = te^t.$$

(There are constants of integration here, but since we just want one particular solution, we can choose them to be zero.) Thus,

$$y = (1+t)u_1 + e^t u_2 = -\frac{1+t}{2}e^{2t} + te^{2t} = \frac{t-1}{2}e^{2t}.$$

J. *Given that the general solution to* $t^2y'' - 4ty' + 4y = 0$ *is* $y = C_1t + C_2t^4$, *solve the following initial value problem:*

$$t^2y'' - 4ty' + 4y = -2t^2, \quad y(1) = 2,\ y'(1) = 0.$$

Although this is not a constant-coefficient problem, since we know the general solution to the associated homogeneous equation we can solve the inhomogeneous equation with the method of variation of parameters.

Put $y = u_1t + u_2t^4$, and assume that

$$u_1't + u_2't^4 = 0.$$

Then

$$\begin{aligned} y' &= u_1 + 4u_2t^3, \\ y'' &= u_1' + 4u_2't^3 + 12u_2t^2. \end{aligned}$$

We find that

$$\begin{aligned} t^2y'' - 4ty' + 4y = 12u_2t^4 - 4u_1t - 16u_2t^4 + 4u_1t + 4u_2t^4 + t^2u_1' + 4t^5u_2' \\ = t^2(u_1' + 4u_2't^3). \end{aligned}$$

Thus,

$$u_1' + 4u_2't^3 = -2,$$

so that $u_1' = -2 - 4u_2't^3$. Substituting this into the assumption equation gives

$$0 = u_1't + u_2't^4 = -2t - 3u_2't^4,$$

or

$$u_2' = -\frac{2}{3}t^{-3}.$$

Thus, $u_2 = \frac{1}{3}t^{-2} + C_2$. Also,

$$u_1' = -2 + \frac{8}{3} = \frac{2}{3},$$

so $u_1 = \frac{2}{3}t + C_1$. So the general solution to the equation is

$$y = C_1t + C_2t^4 + \frac{1}{3}t^2 + \frac{2}{3}t^2 = C_1t + C_2t^4 + t^2.$$

For this general y, $y(1) = C_1 + C_2 + 1$ and $y'(1) = C_1 + 4C_2 + 2$. Solving for C_1 and C_2 gives $C_1 = 2$, $C_2 = -1$. Thus,

$$y = 2t + t^2 - t^4.$$