Week 9 solutions

(Assignment 19 had no hand-graded component.)

ASSIGNMENT 20.

3.7.9. *A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyn·s/cm. If the mass is pulled down an additional 2 cm and then released, find its position u at any time t. Plot u versus t.*

Note: This problem is written in the CGS unit system, which is a rescaling of the SI system of metric units, taking centimeters, grams, and seconds as the fundamental units. A **dyne**, dyn, is the unit of force, equal to 10^{-5} N or $1\,\mathrm{g}\cdot\mathrm{cm/s^2}$. It is fine to convert to SI units, to stick with CGS units, or to use a hybrid of the two, as long as you're consistent and convert where necessary. I will stick with CGS, meaning that $g = 980\,\mathrm{cm/s^2}$.

First, we calculate the spring constant, using the equation $mg = kL$. We get

$$k = mg/L = 20 \cdot 980/5 = 3920\,\mathrm{g/s^2}.$$

The equation for the motion of the spring is then

$$20u'' + 400u' + 3920u = 0,$$

where u is measured in cm below equilibrium, and time is measured in seconds. We can simplify this to

$$u'' + 20u' + 196u = 0.$$

The roots of the characteristic equation are

$$r = -10 \pm 4\sqrt{6}i,$$

so the general (real-valued) solution is

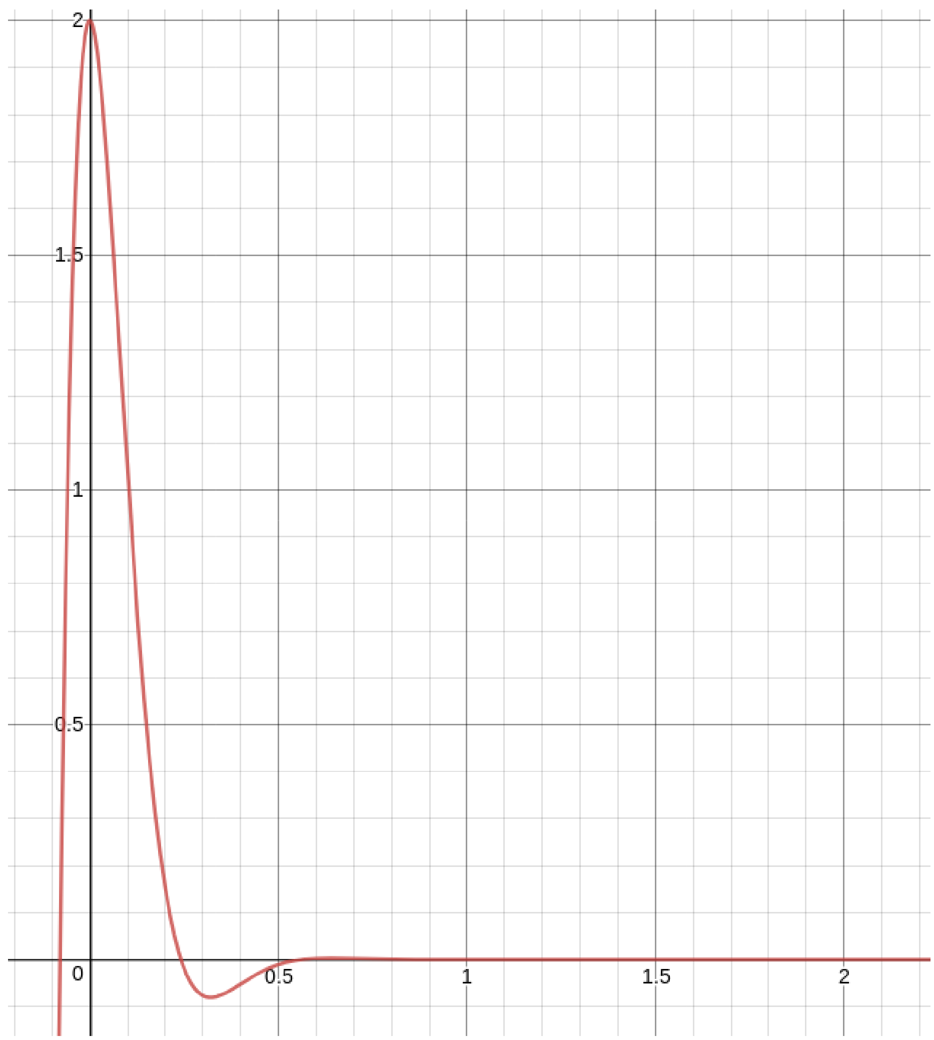
$$u = C_1 e^{-10t}\cos(4\sqrt{6}t) + C_2 e^{-10t}\sin(4\sqrt{6}t)\ [\mathrm{cm}].$$

Note that $4\sqrt{6} \approx 9.798$. We calculate that

$$u' = (-10C_1 + 4\sqrt{6}C_2)e^{-10t}\cos(4\sqrt{6}t) + (-4\sqrt{6}C_1 - 10C_2)e^{-10t}\sin(4\sqrt{6}t).$$

The initial conditions are $u(0) = 2$, $u'(0) = 0$, so $C_1 = 2$, $C_2 = 5/\sqrt{6} \approx 2.041$. So the solution is

$$u = 2e^{-10t}\cos(4\sqrt{6}t) + \frac{5}{\sqrt{6}}e^{-10t}\sin(4\sqrt{6}t).$$



While not overdamped (there are still cosine and sine terms), the spring is very heavily damped, and the oscillations are not really visible on this graph.

Determine the quasi frequency and the quasi period.

The quasi frequency is just the frequency at which the cosine and sine functions appearing in the formula for u oscillate. So it is $4\sqrt{6}\,\mathrm{Hz}$. The quasi period is $2\pi/4\sqrt{6}\,\mathrm{s} \approx 0.6413\,\mathrm{s}$.

Determine the ratio of the quasi period to the period of the corresponding undamped motion.

If the motion were undamped, it would be described by the differential equation

$$u'' + 196u = 0.$$

The roots of the characteristic polynomial are purely imaginary, $r = \pm 14i$, and the general solution is

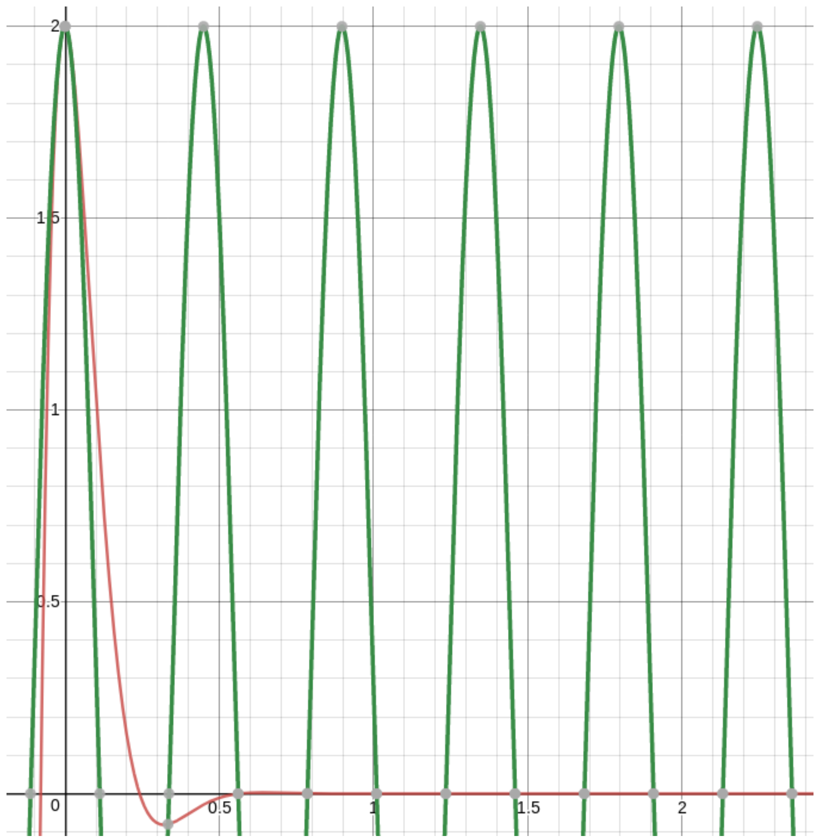
$$u = C_1 \cos(14t) + C_2 \sin(14t).$$

No matter the initial conditions, this oscillates with frequency 14 Hz and period $2\pi/14$.

The ratio of damped quasi period to undamped period is

$$\frac{T_{\text{damped}}}{T_{\text{undamped}}} = \frac{2\pi/4\sqrt{6}}{2\pi/14} = \frac{14}{4\sqrt{6}} \approx 1.429.$$

The undamped spring oscillates quite a bit faster than the damped spring. This graph compares the damped spring with an undamped spring moving with the same initial conditions:



Also find the time τ such that $|u(t)| < 0.05$ cm for all $t > \tau$.

I did this with Desmos. The absolute value of u stays below 0.05 cm after the first time it crosses this line, at $t = 0.2246$.

28. *The position of a certain undamped spring-mass system satisfies the initial value problem*

$$u'' + 2u = 0, \quad u(0) = 0,\ u'(0) = 2.$$

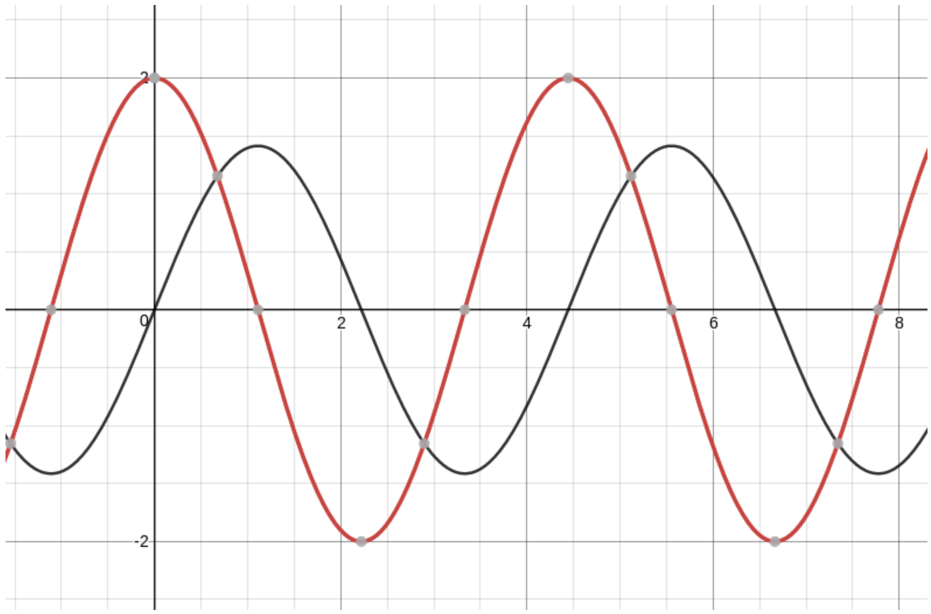
(a) *Find the solution of this initial value problem.*

The characteristic polynomial, r^2+2, has roots $r = \pm\sqrt{2}i$. So the general solution is $u = C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t)$. We have $u(0) = C_1 = 0$ and $u'(0) = \sqrt{2}C_2 = 2$, so the solution to the initial value problem is

$$u = \sqrt{2}\sin(\sqrt{2}t).$$

(b) *Plot u versus t and u' versus t on the same axes.*

Here is a graph:

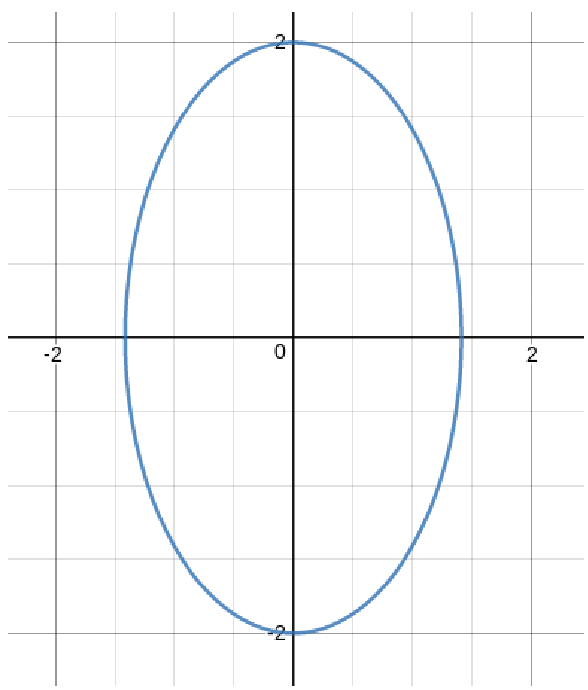


(c) *Plot u' versus u; that is, plot $u(t)$ and $u'(t)$ parametrically with t as the parameter. ... What is the direction of motion in the phase plot as t increases?*

I did this on Desmos by entering the parametric equations:

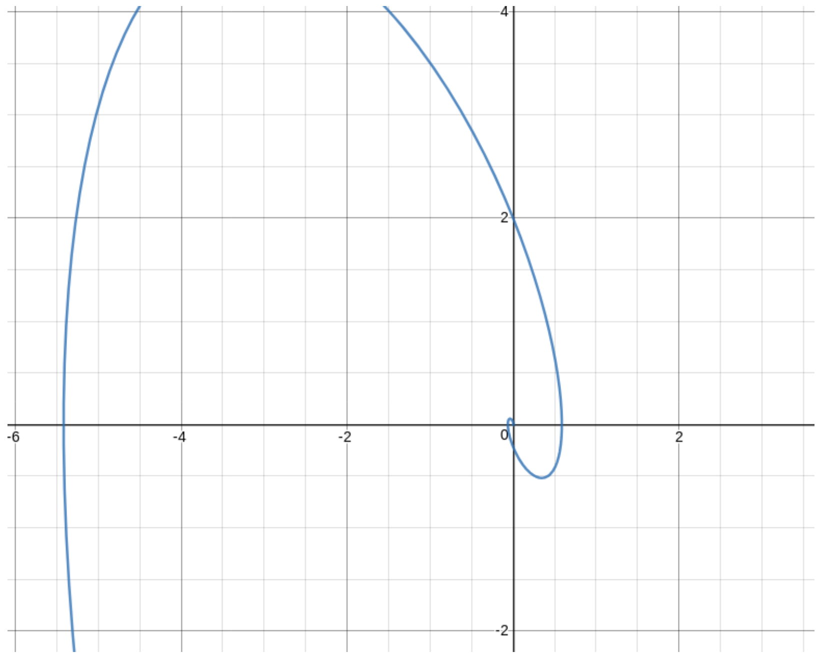
$$(\sqrt{2}\sin(\sqrt{2}t), 2\cos(\sqrt{2}t))$$

and then adjusting the range for t to get the complete, closed curve.



As t increases starting from 0, which corresponds to $(u, u') = (0, 2)$, u, which is a sine function, starts to increase, and u', which is a cosine function, starts to decrease. So the point moves clockwise around the ellipse. (This is with u as the x-axis and u' as the y-axis; if you switch the axes, you get the opposite answer.)

These graphs can be useful ways of visualizing solutions to second-order equations. Here's a similar graph for a damped spring:



ASSIGNMENT 21.

3.8.7. (a) *Find the solution to Problem 5.*

Problem 5 reads:

A mass weighing 4 lb stretches a spring 1.5 in. The mass is given a positive displacement of 2 in from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of 2 cos 3t lb, formulate the initial value problem describing the motion of the mass.

Since the mass weighs 4 lb (a unit of *force*), its mass is

$$m = 4\,\text{lb}/g = 4\,\text{lb}/(32\,\text{ft}/\text{s}^2) = 0.125\,\text{slug}.$$

(A **slug** is an imperial unit of mass, equal to $1\,\text{lb}\cdot\text{s}^2/\text{ft}$.) The spring constant is

$$k = mg/L = 4\,\text{lb}/(1/8\,\text{ft}) = 32\,\text{lb}/\text{ft}.$$

So the initial value problem is

$$0.125u'' + 32u = 2\cos(3t), \quad u(0) = 1/6,\ u'(0) = 0.$$

Here time is measured in seconds and u is measured in feet.

Now we solve the problem. Multiplying through by 8 gives

$$u'' + 256u = 16\cos(3t).$$

The roots of the characteristic equation are $r = \pm 16i$. So the general solution to the associated homogeneous equation,

$$u'' + 256u = 0,$$

is

$$u = C_1 \cos(16t) + C_2 \sin(16t).$$

We solve the inhomogeneous equation using undetermined coefficients. Suppose the solution has the form

$$u = A\cos(3t) + B\sin(3t).$$

Then $u'' = -9A\cos(3t) - 9B\sin(3t)$. Substituting into the inhomogeneous equation gives

$$247A\cos(3t) + 247B\sin(3t) = 16\cos(3t).$$

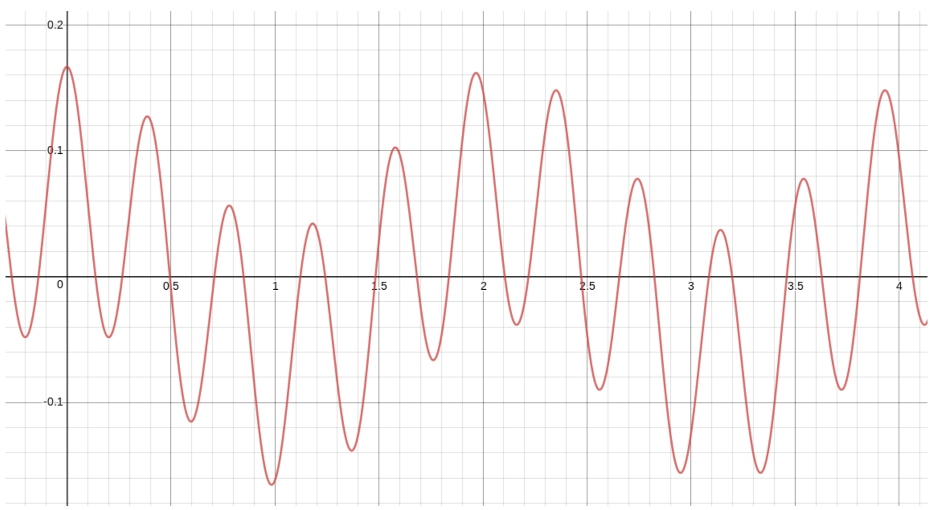
So $B = 0$ and $A = 16/247$. Thus, the general solution to the inhomogeneous equation is

$$u = \frac{16}{247}\cos(3t) + C_1\cos(16t) + C_2\sin(16t).$$

We have $u(0) = 16/247 + C_1$ and $u'(0) = 16C_2$. So $C_2 = 0$ and $C_1 = 1/6 - 16/247 \approx 0.1019\,\text{ft}$. We obtain

$$u = 0.1019\cos(16t) + 0.0648\cos(3t).$$

(b) *Plot the graph of the solution.*



(c) *If the given external force is replaced by a force $4\sin\omega t$ of frequency ω, find the value of ω at which resonance occurs.*

Since the spring is undamped, the resonant frequency is the natural frequency, $\omega = 16$ Hz. Notice that if we were to solve the inhomogeneous equation

$$u'' + 256u = 4\sin(16t)$$

using undetermined coefficients, we would have to get a 'resonant' term of the form $At\cos(16t)$ or $At\sin(16t)$, because $\cos(16t)$ and $\sin(16t)$ are solutions of the associated homogeneous equation. (You should try solving it, and graphing the solution, if you're unconvinced.) If the spring were damped, the resonant frequency would differ slightly from the natural frequency – compare Figure 3.8.2, euqation (14) in section 3.8, and the surrounding discussion in the book.