MA303 review sheet

1 "Chapter 0"

- What is a differential equation? What does it mean for a function to solve a differential equation?
- What does it mean for a differential equation to be linear/nonlinear/homogeneous?
- How do you translate between real-world situations and differential equations?
- You should be comfortable with: complex numbers, integration techniques including integration by parts, trig identities, partial derivatives.
- You should also know how to solve linear homogeneous ODEs with constant coefficients (though this can be done with the techniques from chapter 5).
- ...and how to solve some nonhomogeneous ODEs, particularly with the method of undetermined coefficients.

2 Chapter 5: Linear Systems

- Solve linear systems of differential equations, of the form

$$\mathbf{x}' = A\mathbf{x}.$$

  - Start by finding the eigenvalues and eigenvectors of A: that is, the pairs $(\lambda, \mathbf{v})$ such that
  $$A\mathbf{v} = \lambda\mathbf{v}.$$
  - A **real** eigenvalue corresponds to an **exponential** solution
  $$\mathbf{x} = \mathbf{v}e^{\lambda t}.$$
  - A **complex** eigenvalue works exactly the same way, giving a complex-valued solution. However, if A has real entries, its nonreal eigenvalues occur in conjugate

pairs, and one can find real-valued solutions by taking the real and imaginary part:

$$\mathbf{x}_1 = \mathrm{Re}(\mathbf{v}e^{\lambda t}), \mathbf{x}_2 = \mathrm{Im}(\mathbf{v}e^{\lambda t}).$$

These can be computed using Euler's formula

$$e^{a+ib} = e^a(\cos(b) + i\sin(b)),$$

giving you solutions that are **exponential times trigonometric**.

- **Repeated eigenvalues**:
  (i) If an eigenvalue repeated k times has k linearly independent eigenvectors, then there's nothing to worry about: you can use the above methods to find k linearly independent solutions associated to the eigenvalue.
  (ii) Otherwise, you need to look for generalized eigenvectors, which are solutions to $(A - \lambda I)^r \mathbf{v} = 0$. The set of generalized eigenvectors forms a k-dimensional vector space.
  (iii) Suppose $\mathbf{v}_1$ satisfies $(A - \lambda I)^r \mathbf{v}_1 = 0$ but not $(A - \lambda I)^{r-1}\mathbf{v}_1 = 0$. Let

  $$\mathbf{v}_2 = (A - \lambda I)\mathbf{v}_1, \ldots, \mathbf{v}_r = (A - \lambda I)^{r-1}\mathbf{v}_1.$$

  Then the following is a solution to $\mathbf{x}' = A\mathbf{x}$:

  $$\mathbf{x}(t) = \left(\frac{t^{r-1}}{(r-1)!}\mathbf{v}_r + \cdots + t\mathbf{v}_2 + \mathbf{v}_1\right) e^{\lambda t}.$$

  (iv) In the 2×2 case, let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ. Let $\mathbf{w}$ satisfy

  $$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

  Then two linearly independent solutions are

  $$\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t}, \quad \mathbf{x}_2(t) = (t\mathbf{v} + \mathbf{w})\, e^{\lambda t}.$$

- Understand how to represent solutions to a system graphically, either as a phase plane or as graphs of the various coordinates with respect to time.
- Applications: damped and undamped harmonic oscillators. Two examples we've seen several times are spring-mass systems and RLC circuits.

3 Chapter 6: Nonlinear systems

- The behavior of solutions to linear systems around the origin depends on the eigenvalues. See the table on the next page.

Name	How do the solutions behave?	When does it happen?
Source	Leave the origin as $t \to \infty$	Eigenvalues have real parts > 0
Sink	Approach the origin as $t \to \infty$	EIgenvalues have real parts < 0
Saddle point	Approach the origin along one line, then leave along another	One positive and one negative eigenvalue
Proper node	Approach/leave along all lines through the origin	Repeated nonzero eigenvalue with a full (2-dimensional) space of eigenvectors
Improper node	Approach/leave along one line	Real nonzero eigenvalues and not a proper node
Spiral point	Approach/leave along spirals	Complex eigenvalues with nonzero real parts
Center	Orbit the origin along ellipses	Complex eigenvalues with zero real part
Joker's trick	If zero is an eigenvalue, be careful and keep your wits about you	Zero is an eigenvalue
Stable	Solutions that start close to the origin stay close to the origin	Nodal or spiral sink, or center
Unstable	Not stable	Nodal or spiral source, or saddle point
Asymptotically stable	Solutions that start close to the origin approach the origin as $t \to \infty$	Nodal or spiral sink

- Now consider nonlinear systems of the form

$$\begin{aligned} x' &= F(x,y), \\ y' &= G(x,y). \end{aligned}$$

A **critical point** of the system is a point (x,y) where $x' = y' = 0$. The same sort of analysis can be used to describe behavior close to the critical point.

- The **Jacobian** of the system is the matrix

$$J(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}.$$

- Let (x_c, y_c) be a critical point of the system. We say that the system is **almost linear** there if (i) F and G have continuous first partial derivatives at (x_c, y_c), (ii) (x_c, y_c) is an isolated critical point, and (iii) zero is not an eigenvalue of $J(x_c, y_c)$. In this case, the system has a Taylor expansion

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = J(x_c, y_c) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} r(u,v) \\ s(u,v) \end{pmatrix},$$

where $u = x - x_c$, $v = y - y_c$, and r and s are "remainder" functions satisfying

$$\lim_{(u,v)\to(0,0)} \frac{r(u,v)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)} \frac{s(u,v)}{\sqrt{u^2+v^2}} = 0.$$

The **linearization** of the original system at (x_c, y_c) is the linear system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = J(x_c, y_c) \begin{pmatrix} u \\ v \end{pmatrix}.$$

- If the nonlinear system is almost linear, its linearization approximates it well near the critical point. In particular, the critical point of the nonlinear system is of the same type and stability as the critical point of the linearization *except* in two special cases:

  (i) If the linearization has a center (complex conjugate eigenvalues with zero real part), the nonlinear system can have a center or a stable or unstable spiral point.

  (ii) If the linearization has a node with equal real eigenvalues, the nonlinear system can have a node or a spiral point, but with the same stability as the node in the linearization.

- You should be able to sketch and interpret phase planes (graphs of y versus x) and graphs of x and y versus t.

- Turn higher-order ODEs into first-order systems, and vice versa. This is particularly important for mechanical systems, which are often given as second-order ODEs or systems thereof.

- **Application: interacting species.**

  (a) **The predator-prey model.**

  $$\begin{aligned} x' &= ax - pxy, \\ y' &= -by + qxy, \end{aligned}$$

  where x is the prey population, y is the predator population, and a, b, p, q are positive constants. This has a nonzero critical point at $(b/q, a/p)$, and solutions orbit it stably with angular frequency $\sqrt{ab}$.

  (b) **The competing species model.**

  $$\begin{aligned} x' &= a_1x - b_1x^2 - c_1xy, \\ y' &= a_2y - b_2y^2 - c_2xy, \end{aligned}$$

  where x and y are the populations of the two species and the other numbers are positive constants. The two populations grow logistically on their own but also compete over resources, leading to nesgative effects of their interaction. The system has four critical points: one at the origin, two of the form $(K_x, 0)$ and $(0, K_y)$ at which one species is extinct and the other is at carrying capacity, and a fourth where both species have nonzero population. The fourth critical point is an unstable saddle point if $c_1c_2 > b_1b_2$, and an asymptotically stable node if $c_1c_2 = b_1b_2$. In the unstable case, which species survives and which goes extinct depends on the species' initial values.

  (c) **Other examples.** You should be able to apply this sort of reasoning to other situations. What if the species in (b) cooperate (so that the negative terms $-c_1xy$, $-c_2xy$ are replaced by positive ones)? What if the logistic terms in (b) are added into the predator-prey model of (a)? What if one species is a scavenger that reproduces not based on the other species' population, but rather its death rate? What if there are more than two species forming a food chain or food network?

- **Application: nonlinear mechanics.**

  (a) **Nonlinear springs.**

  $$mx'' = -kx + \beta x^3,$$

  where k is the spring constant, m is the mass, and β is another constant (negative for a soft spring, positive for a hard spring). There could also be damping. In this undamped case, the hard spring still has a single critical point at the origin (a center), while the soft spring has additional saddle points at nonzero values of x. One useful technique was integrating

$$mv\,dv + (kx - \beta x^3)\,dx = 0$$

to find a constant energy value

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 - \frac{1}{4}\beta x^4 = E.$$

(b) **Nonlinear pendulums.**

$$\theta'' + \frac{g}{L}\sin(\theta) = 0,$$

where θ is the angle of the pendulum (chosen so that $\theta = 0$ when the pendulum is at the bottom of its swing). This has a stable critical point at $\theta = 0$ and an unstable one at $\theta = \pi$. Again, there could also be damping.

4 Chapter 9: Fourier series

- Know when functions are periodic, and what their periods are. A period is any nonzero number p such that $f(x+p) = f(x)$ for all p.

- Know when functions are piecewise smooth. A function $f(t)$ is piecewise smooth on an interval $[a, b]$ if we can partition the interval as

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

such that f is continuously differentiable on each subinterval $[t_i, t_{i+1}]$; the one-sided limits

$$f(t_i+) = \lim_{t \to t_i^+} f(t) \quad \text{and} \quad f(t_i-) = \lim_{t \to t_i^-} f(t)$$

are well-defined and finite (at a and b, we only care about the side that's within the interval); and likewise for the one-sided limits of f'.

- A $2L$-periodic function $f(t)$ has a Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

The coefficients are defined by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt.$$

- **Convergence theorem:** if f is periodic and piecewise smooth, then its Fourier series converges to $f(t)$ at each t where f is continuous, and to $\frac{1}{2}(f(t+) + f(t-))$ at each point where f is discontinuous.

- **Differentiation theorem:** if f is periodic, piecewise smooth, piecewise twice differentiable, *and continuous everywhere*, then we can find the Fourier series for f' by differentiating term by term. The key example where this fails is the square wave.

- **Integration theorem:** if f is $2L$-periodic and piecewise continuous with Fourier coefficients a_n and b_n, then

$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{N=1}^{\infty}\left(\frac{a_n L}{n\pi}\sin\left(\frac{n\pi t}{L}\right) - \frac{b_n L}{n\pi}\left(\cos\left(\frac{n\pi t}{L}\right) - 1\right)\right).$$

- If f is **odd** ($f(-x) = -f(x)$) and $2L$-periodic, its Fourier series is a sine series, with

$$a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt.$$

- If f is **even** ($f(-x) = f(x)$) and $2L$-periodic, its Fourier series is a cosine series, with

$$a_n = \frac{2}{L}\int_0^L f(t)\cos(\left(\frac{n\pi t}{L}\right)\,dt.$$

- More generally, you'll find it helpful to be comfortable manipulating integrals of periodic functions, and functions with nice symmetry (like even and odd functions).

- **Application: periodically forced, undamped harmonic oscillators.**

$$mx'' + kx = F(t).$$

  - If $F(t)$ is a simple sine or cosine function, you can solve this with the method of undetermined coefficients. In

$$mx'' + kx = A\cos(\omega t),$$

    look for a solution of the form $x_P = B\cos(\omega t)$ and solve for B. Then the general solution is $x = C_1x_1 + C_2x_2 + x_P$, where $C_1x_1 + C_2x_2$ is the general solution to the associated homogeneous equation

$$mx'' + kx = 0.$$

    x_P is the **steady periodic solution** to the problem.

- If $F(t)$ is a more general piecewise smooth periodic function, such as a square or triangle wave, write F as a Fourier series, do the above for each term of the Fourier series, and combine them. That is, if

$$F(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right),$$

and x_n^A, resp. x_n^B, resp. x_0, is the steady periodic response to the term $a_n \cos\left(\frac{n\pi t}{L}\right)$, resp. $b_n \sin\left(\frac{n\pi t}{L}\right)$, resp. $a_0/2$, then the general solution is

$$x = C_1 x_1 + C_2 x_2 + x_0 + \sum_{n=1}^{\infty} (x_n^A + x_n^B).$$

- The oscillator described by $mx'' + kx$ has a **natural** (angular) **frequency** $\omega_0 = \sqrt{k/m}$. This is the frequency of its unforced oscillations. If $F(t)$ has a Fourier series term with frequency close to ω_0, that term will provoke a large response.
- For a term with frequency exactly equal to ω_0,

$$mx'' + kx = A\cos(\omega_0 t),$$

one instead looks for a solution of the form $x = Bt\sin(\omega_0 t) + Ct\cos(\omega_0 t)$. These solutions are not periodic, but instead have a steadily growing amplitude. This is called **pure resonance**.

- **Application: periodically forced, damped harmonic oscillators.**

$$mx'' + \gamma x' + kx = F(t).$$

This is the same as the previous case, except that the method of undetermined coefficients is more complicated. If the right-hand side is $A\cos(\omega t)$, one generally looks for a steady periodic solution of the form $B\cos(\omega t) + C\sin(\omega t)$. This can also be written as $R\cos(\omega t - \delta)$, meaning that the response is phase-shifted from the forcing function.

5 Chapter 9, part 2: PDEs

- Recognize when a problem consisiting of a PDE with some boundary conditions is linear and homogeneous. This means that, if y_1 and y_2 are solutions to the problem, so is any linear combination $C_1 y_1 + C_2 y_2$.

- **The heat equation in one dimension.**

$$u_t = K u_{xx},$$

where $u(x,t)$ is the temperature distribution on a laterally insulated rod, and K is the thermal diffusivity of the rod. We only solved this in tandem with various **boundary conditions**. One solves it by separating variables. Assume that u is of the form $X(x)T(t)$. Then the heat equation says

$$X(x)T'(t) = KX''(x)T(t),$$

or

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{KT(t)} = -\lambda.$$

Since each side is independent of one of the two variables, they must both be constant. The boundary conditions then give conditions on X, allowing you to specify the values λ such that

$$X'' + \lambda X = 0 \text{ (together with the boundary conditions)}$$

has a nontrivial solution. These values λ_n are the **eigenvalues** of the problem, and the corresponding solutions X_n are the **eigenfunctions**. Then also solve

$$T_n' + \lambda_n K T_n = 0,$$

to get "building block solutions"

$$u_n(x,t) = X_n(x)T_n(t).$$

The general solution is then

$$u = \sum C_n X_n T_n.$$

- What does it mean to say that this is a general solution? It means that if we specify an initial distribution $f(x) = u(x,0)$, there's a unique solution u of this form with initial values $f(x)$. In other words, we can choose C_n so that

  $$f(x) = \sum_{n=1}^{\infty} C_n X_n(x) T_n(0).$$

  Typically, we can do this either by using the theory of Fourier series, or Sturm-Liouville theory (see below).
- Two early cases we looked at: if the boundary conditions say $u(0,t) = u(L,t) = 0$ (the ends are held at temperature zero), then the general solution is

  $$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-Kn^2\pi^2 t}{L^2}\right).$$

  If the ends are held at arbitrary fixed temperatures, you can get the general solution by adding the appropriate linear function of x to the above equation.

- If the boundary conditions say $u_x(0,t) = u_x(L,t) = 0$ (the ends are insulated), then the general solution is

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-Kn^2\pi^2 t}{L^2}\right).$$

- You've seen a few more complicated cases, including some involving heat transfer through the ends, and some where the boundary conditions are different at the two ends.
- You should also mentally prepare yourself for problems like this where there's more than one linearly independent eigenfunction for a single eigenvalue.

- **The wave equation in one dimension.**

$$y_{tt} = v^2 y_{xx}$$

where $y(x,t)$ is the (transverse or longitudinal) displacement of a vibrating medium (such as a string, a pool, or a column of gas), and v is the speed at which waves travel through the medium. This is also solved by separation of variables. If the ends $x = 0$ and $x = L$ are fixed, the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi vt}{L}\right)\right).$$

- We solved this by splitting the general problem into two simpler ones: one where the initial position of the string was zero, and one where the initial velocity was zero. These are each another linear, homogeneous boundary condition, so can be used for separation of variables.
- The above solution describes $y(x,t)$ as a superposition of standing waves. Each standing wave has wavelength $2L/n$, for some n, and frequency (in cycles per second) $nv/(2L)$. The wave speed v is the product of the wavelength and the frequency. Note that all the frequencies are integer multiples of the lowest one, the fundamental frequency $v/(2L)$.
- The **d'Alembert solution**: *any* piecewise twice differentiable function of the form

$$y(x,t) = F(x+vt) + G(x-vt)$$

is a solution to the wave equation. This describes y as a superposition of travelling waves, one moving to the left with speed v and one moving to the right with speed v. If the ends of the string are fixed, the solutions are as above with both F and G odd and $2L$-periodic. In particular, if the ends of the string are fixed, the initial velocity is zero, and the initial displacement is $f(x)$, then writing $F_{\text{odd}}(x)$ for the odd $2L$-periodic extension of f, we have

$$y(x,t) = \frac{1}{2}(F_{\text{odd}}(x+vt) + F_{\text{odd}}(x-vt)).$$

- **Laplace's equation.**
$$\Delta u = 0$$
or
$$u_{xx} + u_{yy} = 0$$
or
$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$
(in polar coordinates). Among other things, solutions to this give steady-state heat distributions, i. e., heat distributions which aren't changing over time.
  - We solved this on the rectangle and the disk, and on your homework you solved it on the washer and on various infinite strips. In all cases, we use separation of variables.
  - In the case of the disk, there were a few new wrinkles. First, all functions $u(r, \theta)$ of polar coordinates (r, θ) satisfy the periodicity condition
$$u(r, \theta) = u(r, \theta + 2\pi).$$
This can be used like a linear, homogeneous boundary condition.
  - Second, some of the building block solutions uncovered by separation of variables go off to infinity at the center of the disk. These solutions can't describe heat distributions, so we throw them out when solving the steady-state heat equation.
  - Third, at one point in the solution we had to solve the **Euler equation**
$$r^2 R'' + r R' - n^2 R = 0.$$
We solved this by looking for solutions of the form $R = r^k$. (If $n = 0$, $R = \ln(r)$ was also a solution.) This technique more generally works for any ODE of the form
$$a r^2 R'' + b r R' + c R = 0.$$

6 Chapter 10: Sturm-Liouville theory and more eigenvalue problems

- A **Sturm-Liouville problem** is a single-variable boundary value problem of the form
$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x) y = 0, \quad (a < x < b),$$
$$\alpha_1 y(a) - \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0,$$
where α_1 and α_2 are not both 0, and β_1 and β_2 are not both 0. The values of λ for which the problem has nontrivial solutions are called **eigenvalues**, and those solutions

are called **eigenfunctions**. The problem is **regular** if p, p', q, and r are continuous on $[a, b]$, and $p > 0$, $r > 0$ on $[a, b]$. It is **nonnegative** if additionally, $q \geq 0$ on $[a, b]$, and α_1, α_2, β_1, $\beta_2 \geq 0$.

- **Eigenvalue theorem**: the eigenvalues of a regular S-L problem are real numbers λ_n with
$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = +\infty.$$
Each eigenvalue has a single associated eigenfunction, up to scalar multiples. If the problem is nonnegative, so are all the eigenvalues. (If 0 is an eigenvalue, we usually call it λ_0.)

- **Orthogonality theorem**: If y_n and y_m are eigenfunctions for the same S-L problem, with $n \neq m$, then
$$\int_a^b y_n(x)y_m(x)r(x)\,dx = 0.$$

- **Convergence theorem**: We define the eigenfunction series of a function $f(x)$ on $[a, b]$ as
$$f(x) \sim \sum c_n y_n(x), \quad \text{where } c_n = \frac{\int_a^b f(x)y_n(x)r(x)\,dx}{\int_a^b y_n(x)^2 r(x)\,dx}.$$
If f is piecewise smooth, then this series converges to $f(x)$ at each point x where f is continuous, and to $\frac{1}{2}(f(x+) + f(x-))$ at each point where f is discontinuous.

- Most importantly: you should know how to use these theorems! They let you find general solutions to more complicated versions of the previous PDEs, such as...

- **Application: the one-dimensional heat equation with heat transfer.**
$$u_t = Ku_{xx}, \quad (0 < x < L),$$
$$u(0, t) = 0, \quad hu(L, t) + u_x(L, t) = 0.$$
The general solution is now not given by a Fourier series, but by an eigenfunction series for a Sturm-Liouville problem.
$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\beta_n}{L}x\right) \exp\left(\frac{\beta_n^2}{L^2}t\right),$$
where β_n is the nth positive solution to $\tan(x) = -x/hL$. Of course, there are many other variants of this that you should be able to handle.

- **Application: vibrating beams.** A frictionless vibrating beam satisfies a fourth-order PDE
$$y_{tt} + a^4 y_{xxxx} = 0,$$

where a is a real constant. If an end is simply supported or hinged, then $y = y_{xx} = 0$ there; if it's clamped, then $y = y_x = 0$ there; and if it's free, then $y_{xx} = y_{xxx} = 0$ there. This can be solved using separation of variables, together with the assumption that the solutions do in fact oscillate with respect to time.

- Alternatively, we can look for **steady periodic solutions** of the form $y = X(x)\cos(\omega t - \delta)$. For example, a beam with simply supported ends and initial position 0 has displacement function given by

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n^2\pi^2 a^2}{L^2}t\right).$$

  In this case, the frequencies of vibration are square multiples of the fundamental frequency.

- **Application: underground temperature oscillations.** (Assuming we talk about it on Tuesday.) The earth satisfies the one-dimensional heat equation

$$u_t = K u_{xx}, \quad x > 0,$$

  where x is the depth below the earth's surface. We can assume that the surface temperature $u(0,t)$ oscillates like $T_0 + A_0\cos(\omega t)$. This has a steady periodic solution of the form

$$u = T_0 + A_0 \exp\left(-\sqrt{\frac{\omega}{2K}}x\right)\cos\left(t - \sqrt{\frac{\omega}{2K}}x\right),$$

  meaning that the seasonal temperature oscillation both decays and phase-shifts as the depth increases.