Math 303 Final

Instructions: *This exam is 2 hour long and has 15 questions, worth 200 points in total. No calculators or notes will be permitted.*

If you want your work graded, make sure it's understandable and it's clear which question it's referring to. If you tear pages out, write your name on top of them. If you finish early, you can hand the exam in up front and leave.

Name: Answer Key

Section (circle one): 9-10:15 10:30-11:45

Part I: Multiple Choice

Each question is worth 5 points, and has a single correct answer. There will be no partial credit.

1. A 0.5 kg mass is attached to a spring with spring constant 1 kg/s^2 and damping constant 2 kg/s. The spring is stretched by 0.1 m and released. How does the mass move over time?

   (a) The mass oscillates forever with the same amplitude.

   (b) The mass oscillates with amplitude that increases over time.

   (c) The mass oscillates with amplitude that decreases over time.

   (d) The mass moves back towards equilibrium without oscillating.

   (e) None of the above.

$$0.5x'' + 2x' + x = 0.$$

Characteristic equation: $0.5r^2 + 2r + 1 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 0.5}}{2 \cdot 0.5} = -2 \pm \sqrt{2}$$

Two negative, real solutions so the general solution is:

$$x = C_1 e^{(-2+\sqrt{2})t} + C_2 e^{(-2-\sqrt{2})t}$$

which does not oscillate.

2. Let $f(x)$ be the 4-periodic function such that $f(x) = x^2$ for $-2 \leq x \leq 2$. The Fourier series of f is

$$f(x) \sim \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16\cos(\pi n)}{\pi^2 n^2} \cos\left(\frac{\pi n x}{2}\right).$$

What is the value of

$$\frac{4}{3} + \sum_{n=1}^{\infty} \frac{16\cos(\pi n)}{\pi^2 n^2} \cos\left(\frac{\pi n}{2}\right)?$$

(a) -1

(b) 0

(c) 1

(d) 2

(e) 4

Write $F(x)$ for the sum of the Fourier series.

Then the number we want is $F(1)$.

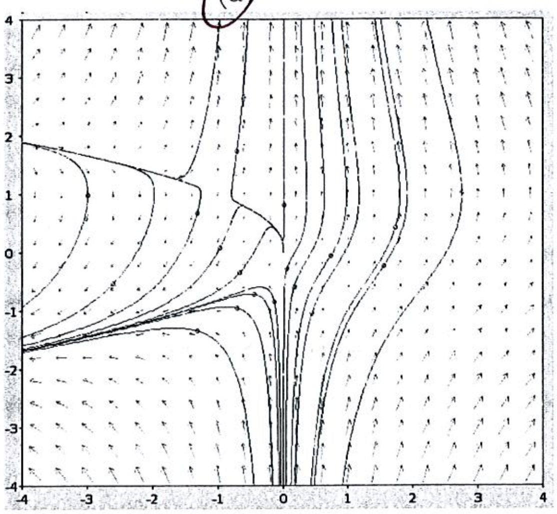
By the convergence theorem (since f is piecewise smooth), $F(x) = f(x)$ wherever $f(x)$ is continuous.

So $F(1) = f(1) = 1$.

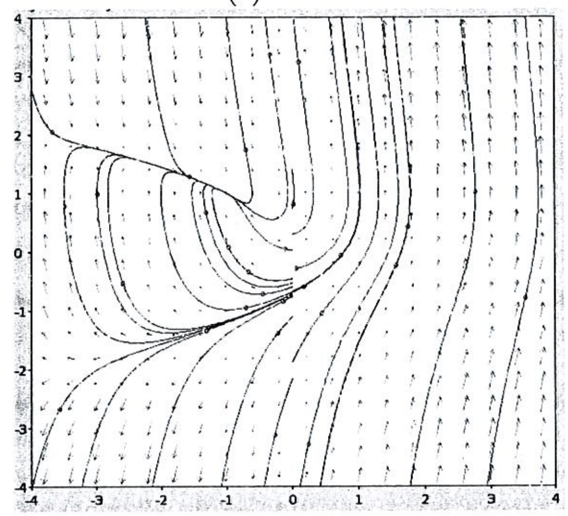
3. Which of the following is the phase plane of the system

$$x' = x - xy,$$
$$y' = 2x + 2y^2?$$

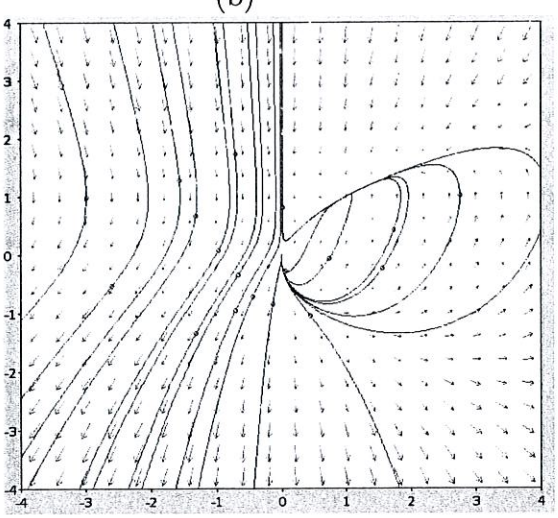
(a)



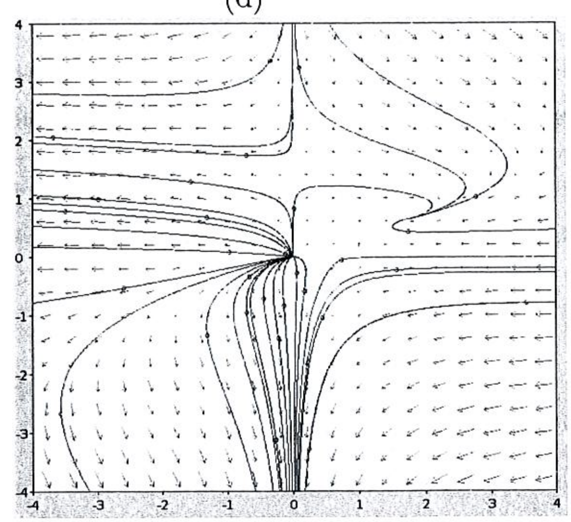
(c)



(b)



(d)



Critical points at: $0 = x - xy = -x(y-1)$

$0 = 2x + 2y^2$

So $x=0,\ y=0$ or $y=1,\ x=-1$.

(a) and (c) appear to have critical points at $(-1,1)$.
However, (c) seems to have non-isolated critical points along the y-axis. (There are other ways to do this, too.)

4. Which of the following problems, consisting of a partial differential equation with some boundary conditions, satisfies the following property?

(P) If u_1 and u_2 are solutions to the problem, then so is any linear combination $C_1u_1 + C_2u_2$.

(a) $u_{xx} + u_{yy} + u_{zz} = 0$ for $x \geq 0$, $\underline{u(0, y, z) = e^{-(y^2+z^2)}}$, u is bounded.

(b) $\underline{r^2u_{rr}^2 + ru_r^2 + u_{\theta\theta}^2} = 0$ for $0 \leq r \leq 1$; $u(r, \theta) = u(r, \theta + 2\pi)$; $u_r(1, \theta) = 0$.

(c) $u_t = Ku_{xx}$ for $0 \leq x \leq 3$; $hu(0, t) - u_x(0, t) = hu(L, t) + u_x(L, t) = 0$.

(d) $y_{tt} + a^4y_{xxxx} = 0$ for $0 \leq x \leq L$; $y(0, t) = y_{xx}(0, t) = y(1, t) = 0$, $\underline{y_{xx}(1, t) = 1}$.

(c) is the only linear homogeneous problem. The parts of the other problems that fail (P) are underlined.

5. Suppose that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$$

is an almost linear system with a critical point at $(1,1)$, and that the Jacobian of the system at $(1,1)$ has two negative eigenvalues. What sort of critical point is $(1,1)$?

(a) Nodal sink

(b) Center

(c) Spiral sink

(d) (a) or (b)

(e) (a) or (c)

If the eigenvalues are distinct, the point is a node. If they're equal, it's a node or spiral. Either way, it's a sink because both eigenvalues are negative.

6. An undamped spring-mass system with mass 1 kg and spring constant 4 kg/s^2 is forced by the sawtooth wave

$$F(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n} \cancel{\sin(nx).}$$

$\sin(nt)$.

What happens to the system over time?

(a) The spring oscillates with constant amplitude and period π.

(b) The spring oscillates with constant amplitude and period 2π.

(c) The spring oscillates with an amplitude that increases over time.

(d) The spring oscillates with an amplitude that decreases over time.

(e) The spring does not move.

$$x'' + 4x = F(t)$$

The natural frequency is $\sqrt{\frac{k}{m}} = 2$.

So the term $-\frac{1}{2}\sin(2t)$ causes resonance.

7. The displacement function of a metal bar with one end clamped and one end free satisfies the following differential equation and boundary conditions:

$$y_{tt} + a^4 y_{xxxx} = 0,$$
$$y(0,t) = y_{xx}(0,t) = 0,$$
$$y_{xx}(L,t) = y_{xxx}(L,t) = 0.$$

Suppose that the bar oscillates steadily and periodically, so that

$$y(x,t) = X(x)\cos(\omega t - \delta).$$

What is the boundary value problem satisfied by X?

(a) $X^{(4)} - \omega^2 X = 0$, $X(0) = X''(0) = X''(L) = X^{(3)}(L) = 0$.
(b) $X'' - a^2\omega^2 X = X'' + a^2\omega^2 X = 0$, $X(0) = X''(L) = 0$.
(c) $X^{(4)} - \frac{\omega^2}{a^4} X = 0$, $X(0) = X(L) = X''(0) = X''(L) = 0$.
(d) $X^{(4)} - \frac{\omega^2}{a^4} X = 0$, $X(0) = X''(0) = X''(L) = X^{(3)}(L) = 0$.
(e) $X^{(4)} - \omega^2 X = 0$, $X(0) = X^{(3)}(0) = X''(L) = X^{(3)}(L) = 0$.

$$y_{tt} + a^4 y_{xxxx} = 0$$

$$X(x)(-\omega^2\cos(\omega t-\delta)) + a^4 X^{(4)}(x)\cos(\omega t-\delta) = 0$$

$$X^{(4)} - \frac{\omega^2}{a^4} X = 0.$$

The boundary conditions all imply things about X.

For example, $y_{xx}(0,t) = 0 \Rightarrow X''(0)\cos(\omega t-\delta) = 0$
$\Rightarrow X''(0) = 0$

8. Vibrations of air in a pipe of length L which is open at both ends are described by the equation

$$y(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi v t}{L} - \delta_n\right).$$

Note that $y_x = 0$ at both ends of the pipe – this is true wherever the pipe is open to the surrounding air.

Suppose that a hole is drilled at $x = L/2$. How does this affect its fundamental frequency?

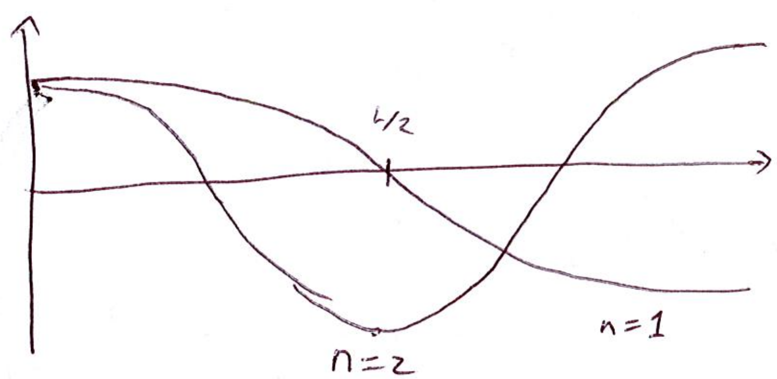
(a) The new fundamental frequency is twice the old fundamental frequency.

(b) The new fundamental frequency is the same as the old fundamental frequency.

(c) The new fundamental frequency is half the old fundamental frequency.

(d) The new fundamental frequency is 1.5 times the old fundamental frequency.

(e) None of the above.



The new lowest-frequency standing wave is n=2, which has twice the old lowest frequency.

9. Consider the Sturm-Liouville problem given by

$$y''(x) + \lambda y(x) = 0 \quad (0 \leq x \leq L)$$

together with one of the following sets of boundary conditions. Under which one of the sets of boundary conditions does the problem have a negative eigenvalue?

(a) $y(0) = y(L) = 0$.
(b) $y'(0) = y'(L) = 0$.
(c) $y(0) = y'(L) = 0$.
(d) $y(0) = 0$, $y'(L) = 2y(L)$.
(e) $y'(0) = 2y(0)$, $y(L) = 0$.

(d) is the only problem that isn't nonnegative.

Alternatively, we can just find the negative eigenvalues of each problem. If $\lambda < 0$, the solutions to

$$y'' + \lambda y = 0 \quad \text{are}$$

$$y = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

So for (d), $y(0) = 0 \Rightarrow 0 = C_1 + C_2 \Rightarrow C_2 = -C_1$

$$y'(L) = 2y(L) \Rightarrow \sqrt{-\lambda}C_1 e^{\sqrt{-\lambda}L} + \sqrt{-\lambda}C_1 e^{-\sqrt{-\lambda}L} = 2C_1 e^{\sqrt{-\lambda}L} - 2C_1 e^{-\sqrt{-\lambda}L}$$

$$(\sqrt{-\lambda} - 2)e^{\sqrt{-\lambda}L} = (-\sqrt{-\lambda} - 2)e^{-\sqrt{-\lambda}L}$$

$$e^{2\sqrt{-\lambda}L} = -\frac{\sqrt{-\lambda}+2}{\sqrt{-\lambda}-2}$$

So $\sqrt{-\lambda} > 0$ is a solution to $e^{2xL} = -\dfrac{x+2}{x-2}$

NOTE: Whether this has a solution depends on L! But the other choices can be checked to not have a negative eigenvalue, no matter what L is.

10. The following are all equations of motion of mechanical systems. Which system has an unstable equilibrium at $x = 1$?

   (a) $x'' + x - x^3 = 0$
   (b) $x'' + x + x^3 = 0$
   (c) $x'' + x' + x - 2x^3 = 0$
   (d) $x'' + \sin(x) = 0$
   (e) $x'' - \sin(x) = 0$

Plug in $x=1$, $x'=0$. Then (a) is the only system giving $x''=0$ In the other equations, the system will accelerate there.

Part II: Long Answer Problems

Each of these questions is worth 30 points. You can get partial credit on these, based on the work you do towards an answer. To get the most partial credit, make sure your work is legible and understandable.

11. Find all the critical points of the following nonlinear system, and describe the type and stability of each critical point as fully as you can.

$$x' = xy - x,$$
$$y' = xy - 2y$$

$0 = xy - x = x(y-1) \Rightarrow x = 0$ or $y = 1$

$0 = xy - 2y = y(x-2) \Rightarrow y = 0$ or $x = 2$.

So the critical points are $(0,0)$ and $(2,1)$.

The Jacobian is $\begin{pmatrix} y-1 & x \\ y & x-2 \end{pmatrix}$.

At $(0,0)$: $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$. Eigenvalues are -1 and -2, which are negative, real, and distinct. So this is a nodal sink. (stable)

At $(2,1)$: $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. $\begin{vmatrix} -\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 2$, $\lambda = \pm\sqrt{2}$.

These are real and have opposite signs, so this critical point is a saddle point (unstable).

12. Find the Fourier series of the even 6-periodic function which is given on the interval $[0,3]$ by

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 0 & 2 \le t < 3. \end{cases}$$

This is even, so it just has a cosine series.

$$L = \frac{\text{period}}{2} = 3.$$

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{3}\int_0^3 f(t)\,dt = \frac{2}{3}\int_1^2 1\,dt = \frac{2}{3}$$

For $n > 0$,

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{2}{3}\int_0^3 f(t)\cos\left(\frac{n\pi t}{3}\right)dt$$

$$= \frac{2}{3}\int_1^2 \cos\left(\frac{n\pi t}{3}\right)dt = \frac{2}{3}\left[\frac{3}{n\pi}\sin\left(\frac{n\pi t}{3}\right)\right]_1^2$$

$$= \frac{2}{n\pi}\left(\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right)$$

So $\boxed{f(t) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right)\cos\left(\frac{n\pi t}{3}\right).}$

13. Find all the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y''(x) + \lambda y = 0 \quad (0 < x < L),$$
$$y'(0) = y(L) = 0.$$

The problem is nonnegative, so we don't have to consider $\lambda < 0$. We do have to consider $\lambda = 0$.

$\underline{\lambda = 0}$: $y = C_1 + C_2 x$

$0 = y'(0) = C_2$

$0 = y(L) = C_1 + C_2 L$, so $C_1 = C_2 = 0$, so $\lambda = 0$ is not an eigenvalue.

$\underline{\lambda > 0}$: $y = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$.

$0 = y'(0) = \sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda} \cdot 0) + \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda} \cdot 0) = \sqrt{\lambda}\, C_2$.

So $C_2 = 0$, $y = C_1 \cos(\sqrt{\lambda}\, x)$

$0 = y(L) = C_1 \cos(\sqrt{\lambda} L)$. So $\sqrt{\lambda} L = \frac{n\pi}{2}$, for odd n.

$$\lambda_n = \frac{n^2 \pi^2}{4L^2} \quad (n \text{ odd} \geq 1)$$

$$y_n = C \cdot \cos\left(\frac{n\pi}{2L} x\right) \quad (n \text{ odd} \geq 1).$$

(Note there are other ways of writing this, e.g.

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{4L^2} \quad \text{for } n = 1, 2, 3, \ldots$$

Also, it's fine to just write $y_n = \cos\left(\frac{n\pi}{2L} x\right)$ for n odd

— this is a basis for the 1-dimensional space of eigenfunctions with eigenvalue λ_n.

14. The general solution to the steady-state heat equation on a washer is, in polar coordinates,

$$B_n r^n \sin(n\vartheta)$$

$$u(r,\theta) = A_0 + B_0 \ln(r) + \sum_{n=1}^{\infty} \left(A_n r^n \cos(n\theta) + \cancel{B_n r^n \sin(\theta)} + C_n r^{-n}\cos(n\theta) + D_n r^{-n}\sin(n\theta)\right).$$

Suppose that the inner radius of the washer is 1 and the outer radius is 2. Find the steady-state temperature subject to the boundary conditions,

$$u(1,\theta) = 0,$$

$$u(2,\theta) = \begin{cases} 1 & 0 \le \theta < \pi \\ -1 & \pi \le \theta < 2\pi. \end{cases}$$

$$u(1,\vartheta) = 0 = A_0 + \sum \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta) + C_n \cos(n\vartheta) + D_n \sin(n\vartheta)\right)$$

For the other boundary, we need to find the Fourier series of the given function, $f(\vartheta) = \begin{cases} 1 & 0 \le \vartheta < \pi \\ -1 & \pi \le \vartheta < 2\pi \end{cases}$ + 2π-periodic.

$f(\vartheta)$ is odd, so this will be a sine series.

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin\left(\frac{n\pi\vartheta}{\pi}\right) d\vartheta = \frac{2}{\pi}\left[-\frac{\cos(n\vartheta)}{n}\right]_0^{\pi} = \frac{2}{n\pi}(1-\cos(n\pi))$$

So: $$u(2,\vartheta) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos(n\pi))\sin(n\vartheta) = A_0 + B_0 \ln(2) + \sum\left(A_n \cdot 2^n \cos(n\vartheta) + B_n \cdot 2^n \sin(n\vartheta) + C_n \cdot 2^{-n}\cos(n\vartheta) + D_n \cdot 2^{-n}\sin(n\vartheta)\right)$$

Comparing coefficients:

$0 = A_0$

$0 = A_0 + B_0 \ln(2)$

so $A_0 = B_0 = 0$

$0 = A_n + C_n \quad (n>0)$

$0 = 2^n A_n + 2^{-n} C_n \quad (n>0)$

so $A_n = C_n = 0$

$\cancel{0 = B_n + D_n \quad (n>0)}$

$\cancel{0 = 2^n B_n + 2^{-n} D_n \quad (n>0)}$

$\cancel{D_n = -B_n}$ ~~so~~

$\cancel{0 = 2^n B_n - 2^{-n}}$

$$0 = B_n + D_n \quad (n > 0) \quad \Rightarrow \quad B_n = -D_n$$

$$\frac{2}{n\pi}(1-\cos(n\pi)) = 2^n B_n + 2^{-n} D_n \quad (n > 0)$$

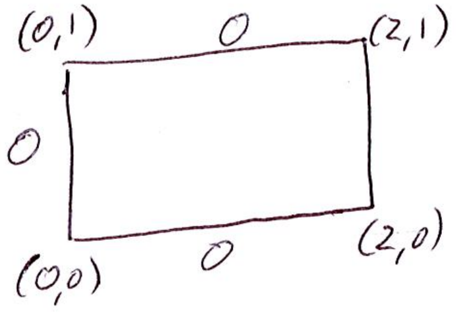
$$= -2^n D_n + 2^{-n} D_n$$

$$D_n = \frac{2(1-\cos(n\pi))}{n\pi(-2^n+2^{-n})} \qquad B_n = \frac{2(1-\cos(n\pi))}{n\pi(2^n - 2^{-n})}$$

Thus, $\boxed{u(r,\theta) = \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n\pi(2^n - 2^{-n})}\left(r^n \sin(n\theta) - r^{-n}\sin(n\theta)\right)}$

15. Find the general solution to the following problem:

$$u_{xx} + u_{yy} = 0 \quad (0 \le x \le 2, 0 \le y \le 1),$$
$$u(x,0) = u(x,1) = 0 \quad (0 \le x \le 2),$$
$$u(0,y) = 0 \quad (0 \le y \le 1).$$



Suppose $u = X(x) \cdot Y(y)$.

Then $X''Y + Y''X = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

The boundary conditions imply $X(0) = Y(0) = Y(1) = 0$.

Start with Y: $Y'' + \lambda Y = 0$, $Y(0) = Y(1) = 0$.

This is a nonnegative S-L problem, so we don't need to check $\lambda < 0$.

$\underline{\lambda = 0}$: $Y = C_1 + C_2 y$

$0 = Y(0) = C_1$

$0 = Y(1) = C_1 + C_2$.

So $C_1 = C_2 = 0$, $\lambda = 0$ is not an eigenvalue.

$\underline{\lambda > 0}$: $Y = C_1 \cos(\sqrt{\lambda} y) + C_2 \sin(\sqrt{\lambda} y)$

$0 = Y(0) = C_1$, $Y = C_2 \sin(\sqrt{\lambda} y)$

$0 = Y(1) = C_2 \sin(\sqrt{\lambda})$. So $\sqrt{\lambda} = n\pi$.

Write $\lambda_n = n^2\pi^2$, $Y_n = \sin(n\pi y)$.

$X_n'' - \lambda_n X_n = 0$, $X_n = C_1 e^{\sqrt{\lambda_n} x} + C_2 e^{-\sqrt{\lambda_n} x} = C_1 e^{n\pi x} + C_2 e^{-n\pi x}$.

$0 = X_n(0) = C_1 + C_2$. Take $C_1 = \frac{1}{2}$. Then

$X_n = \frac{1}{2}(e^{n\pi x} - e^{-n\pi x}) = \sinh(n\pi x)$.

(Note, it's fine to leave X in the "exponential" form, or to take $C_1 = 1$ and get $X = e^{n\pi x} - e^{-n\pi x}$).

The general solution is

$$u = \sum_{n=1}^{\infty} C_n X_n Y_n = \sum_{n=1}^{\infty} C_n \sinh(n\pi x)\sin(n\pi y).$$