Math 303, Homework 10

Due November 14, 2019

1. Here are some problems consisting of partial differential equations with various boundary conditions. For each such problem, say whether or not each of these problems has the following property (P), and briefly explain why.

   (P) If u_1 and u_2 are solutions to the problem, then so is $C_1u_1 + C_2u_2$, for any constants C_1 and C_2.

   (a) $u(x, y, z)$ is a function on the cube $\{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$, such that

   $$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= 0, \\ u_z(x, y, 0) - 3u(x, y, 0) &= 0, \\ u_x(0, y, z) = u_x(1, y, z) &= 0, \\ u_y(x, 0, z) = u_y(x, 1, z) &= 0. \end{aligned}$$

   (b) $u(r, \theta)$ is a function on the unit disk $\{(r, \theta) : r \le 1\}$, such that $r^2u_{rr} + ru_r + u_{\theta\theta} = 0$, and such that u is zero on the bottom semicircle of the boundary, $\{(r, \theta) : r = 1, \pi \le \theta < 2\pi\}$.

   (c) $u(x, y)$ is a function on the rectangle $\{(x, y) : 0 \le x \le a, 0 \le y \le b\}$, such that $u_{xx} + u_{yy} = 0$, and $u(x, 0) = 0$, $u(x, b) = 1$.

2. Waves in air inside a pipe work similarly to waves on a string. Let $y(x, t)$ be the longitudinal displacement of each "layer" of air from its equilibrium position. Then y satisfies the wave equation. If an end of the pipe is closed, then the air can't move at that end, so $y = 0$ at that end. If an end of the pipe is open, then the pressure at that end must be equal to atmospheric pressure, and this turns out to mean that $y_x = 0$ at that end.

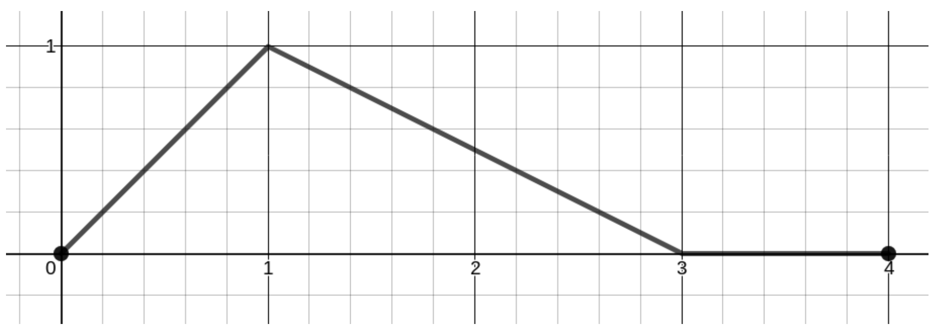
   (a) We can model a simple flute of length L (with no finger-holes) as a pipe with one open end and one closed end. The air is initially non-displaced, and then the flutist blows across the mouth-hole, giving it some initial velocity. In other words, the function $y(x,t)$ is a solution to the problem

$$\begin{aligned} y_{tt} &= v^2 y_{xx}, \\ y(0,t) = y_x(L,t) &= 0, \\ y(x,0) &= 0. \end{aligned}$$

Using the method of separation of variables, find a general formula for $y(x,t)$.

(b) The flute is built so that its fundamental frequency – the lowest frequency appearing in the above series, which will typically also be the loudest – is 432 Hz. This is done to provide the flute with cosmic healing powers. A hole is then drilled at $x = L/3$, so that solutions also have to satisfy $y_x(L/3,t) = 0$. What is the new fundamental frequency? Explain your reasoning.

3. Here's the graph of the position of a string with fixed ends at $t = 0$:



Sketch graphs of the position of the string at $t = 1$, $t = 2$, and $t = 3$. The length of the string is 4, the wave speed is 1, and the initial velocity is zero at each point.