Math 303, Homework 10 solutions

November 30, 2019

1. *Here are some problems consisting of partial differential equations with various boundary conditions. For each such problem, say whether or not each of these problems has the following property (P), and briefly explain why.*

   (P) If u_1 and u_2 are solutions to the problem, then so is $C_1u_1+C_2u_2$, for any constants C_1 and C_2.

   Note: as we've talked about in class, property (P) follows from the problem (the differential equation, and all the boundary conditions) being linear and homogeneous! So if you can recognize this condition easily, it's fine to just say "this problem is linear and homogeneous". The point of the problem is that, in practice, it's often easier to check (P) than it is to prove a problem is linear and homogeneous.

   (a) *$u(x,y,z)$ is a function on the cube $\{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$, such that*

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= 0, \\ u_z(x,y,0) - 3u(x,y,0) &= 0, \\ u_x(0,y,z) = u_x(1,y,z) &= 0, \\ u_y(x,0,z) = u_y(x,1,z) &= 0. \end{aligned}$$

   Suppose that $u = C_1u_1 + C_2u_2$, where u_1 and u_2 are solutions to the problem. Then

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= C_1(u_1)_{xx} + C_2(u_2)_{xx} + C_1(u_1)_{yy} + C_2(u_2)_{yy} + C_1(u_1)_{zz} + C_2(u_2)_{zz} \\ &= C_1((u_1)_{xx} + (u_1)_{yy} + (u_1)_{zz}) + C_2((u_2)_{xx} + (u_2)_{yy} + (u_2)_{zz}) \\ &= 0 + 0 = 0, \end{aligned}$$

   where for the last line we've used the fact that u_1 and u_2 solve the PDE. Next,

$$\begin{aligned} u_z(x,y,0) - 3u(x,y,0) &= C_1(u_1)_z(x,y,0) + C_2(u_2)_z(x,y,0) - 3C_1u_1(x,y,0) - 3C_2u_2(x,y,0) \\ &= C_1((u_1)_z(x,y,0) - 3u_1(x,y,0)) + C_2((u_2)_z(x,y,0) - 3u_2(x,y,0)) \\ &= 0 + 0 = 0, \end{aligned}$$

using the fact that u_1 and u_2 both satisfy this boundary condition. Next,

$$u_x(0, y, z) = C_1(u_1)_x(0, y, z) + C_2(u_2)_x(0, y, z) = C_1 \cdot 0 + C_2 \cdot 0 = 0.$$

The other three conditions are very similar to this one. So this problem satisfies (P).

(b) *$u(r, \theta)$ is a function on the unit disk $\{(r, \theta) : r \leq 1\}$, such that $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$, and such that u is zero on the bottom semicircle of the boundary, $\{(r, \theta) : r = 1, \pi \leq \theta < 2\pi\}$.*

As before, let $u = C_1 u_1 + C_2 u_2$, where u_1 and u_2 are solutions. Then

$$\begin{aligned} r^2 u_{rr} + r u_r + u_{\theta\theta} &= r^2(C_1(u_1)_{rr} + C_2(u_2)_{rr}) + r(C_1(u_1)_r + C_2(u_2)_r) + C_1(u_1)_{\theta\theta} + C_2(u_2)_{\theta\theta} \\ &= C_1(r^2(u_1)_{rr} + r(u_1)_r + (u_1)_{\theta\theta}) + C_2(r^2(u_2)_{rr} + r(u_2)_r + (u_2)_{\theta\theta}) \\ &= 0 + 0 = 0. \end{aligned}$$

This uses the fact that u_1 and u_2 satisfy the PDE. (By the way, notice that this PDE is linear and homogeneous, even though it has nonconstant functions of r and θ, like r^2, appearing in it – what matters is that u and the derivatives of u only appear linearly.)

The boundary condition can be rewritten as $u(1, \theta) = 0$ for $\pi \leq \theta < 2\pi$. Suppose that this is true for u_1 and u_2. Then, for $\pi \leq \theta < 2\pi$,

$$u(1, \theta) = C_1 u_1(1, \theta) + C_2 u_2(1, \theta) = 0 + 0 = 0.$$

(When we talked about the steady-state heat equation on the disk in class, we noticed that functions that take polar coordinates as input automatically satisfy a periodicity condition that's useful for solving PDEs, $u(r, \theta + 2\pi) = u(r, \theta)$. Can you see why this condition also satisfies property (P)?)

(c) *$u(x, y)$ is a function on the rectangle $\{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$, such that $u_{xx} + u_{yy} = 0$, and $u(x, 0) = 0$, $u(x, b) = 1$.*

In this case, the differential equation and the first boundary condition satisfy (P), but the second does not. Indeed, suppose that u_1 and u_2 are solutions to the problem, and let $u = C_1 u_1 + C_2 u_2$. Then for $0 \leq x \leq a$,

$$u(x, b) = C_1 u_1(x, b) + C_2 u_2(x, b) = C_1 + C_2,$$

which might not be 1. *(It's interesting to note that some linear combinations still do work – if u_1 and u_2 are solutions, then any function of the form $C_1 u_1 + (1 - C_1) u_1$ is still a solution. Sometimes, observations like these can be used to handle nonhomogeneous boundary conditions.)*

2. *Waves in air inside a pipe work similarly to waves on a string. Let $y(x, t)$ be the longitudinal displacement of each "layer" of air from its equilibrium position. Then y*

satisfies the wave equation. If an end of the pipe is closed, then the air can't move at that end, so $y = 0$ at that end. If an end of the pipe is open, then the pressure at that end must be equal to atmospheric pressure, and this turns out to mean that $y_x = 0$ at that end.

(a) *We can model a simple flute of length L (with no finger-holes) as a pipe with one open end and one closed end. The air is initially non-displaced, and then the flutist blows across the mouth-hole, giving it some initial velocity. In other words, the function $y(x,t)$ is a solution to the problem*

$$\begin{aligned} y_{tt} &= v^2 y_{xx}, \\ y(0,t) = y_x(L,t) &= 0, \\ y(x,0) &= 0. \end{aligned}$$

Using the method of separation of variables, find a general formula for $y(x,t)$.

Suppose that $y(x,t) = X(x)T(t)$. Then the differential equation is

$$\begin{aligned} XT'' &= v^2 X'' T \\ \frac{T''}{v^2 T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

where for the second line we note that $T''/(v^2T)$ is independent of x, X''/X is independent of t, so they must both be equal to the same constant. The boundary conditions imply

$$X(0)T(t) = X'(L)T(t) = 0,$$

so since $T(t) = 0$ gives the trivial solution, we must have $X(0) = X'(L) = 0$. So we are considering the one-variable boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = X'(L) = 0.$$

This is exactly the boundary value problem that arose from separation of variables in problem 2(a) of homework 9. So we have

$$\lambda_n = \frac{n^2\pi^2}{4L^2}, \quad X_n = \sin\left(\frac{n\pi x}{2L}\right), \quad n \text{ odd} \geq 1.$$

The initial condition $y(x,0) = 0$ implies

$$X(x)T(0) = 0,$$

so $T(0) = 0$ because $X(x) = 0$ would give the trivial solution. Thus, T solves the one-variable problem

$$T'' + v^2\lambda T = 0, \quad T(0) = 0.$$

Solutions to this take the form

$$T = C \sin\left(v\sqrt{\lambda}t\right),$$

so taking $\lambda = \lambda_n$ as above (and $C = 1$) we get

$$T_n = \sin\left(\frac{vn\pi t}{2L}\right), \quad n \text{ odd} \geq 1.$$

The solutions we have obtained with separation of variables are thus

$$u_n = T_n X_n = \sin\left(\frac{n\pi x}{2L}\right)\sin\left(\frac{vn\pi t}{2L}\right), \quad n \text{ odd} \geq 1.$$

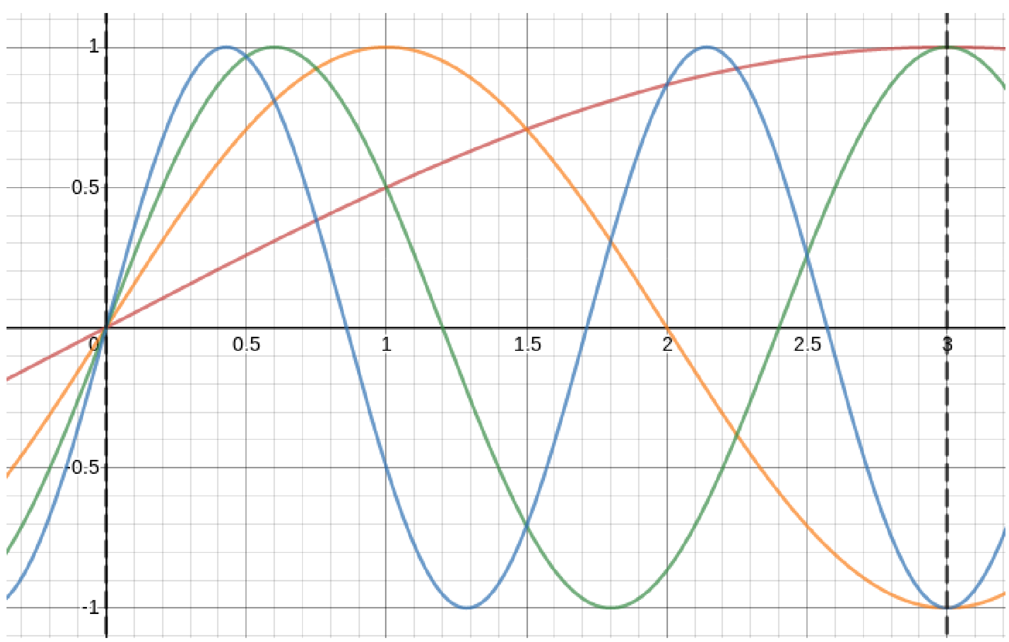
The general solution is

$$u(x,t) = \sum_{n \text{ odd}\geq 1} C_n \sin\left(\frac{n\pi x}{2L}\right)\sin\left(\frac{vn\pi t}{2L}\right). \tag{1}$$

(Can you use your work in homework 9 to convince yourself that this is in fact a general solution, in the sense that, no matter what initial velocity distribution we specify, we can find a solution u of the form (1) with that initial velocity?)

(b) *The flute is built so that its fundamental frequency – the lowest frequency appearing in the above series, which will typically also be the loudest – is 432 Hz. This is done to provide the flute with cosmic healing powers. A hole is then drilled at $x = L/3$, so that solutions also have to satisfy $y_x(L/3, t) = 0$. What is the new fundamental frequency? Explain your reasoning.*

The formula (1) writes the general solution as a linear combination of standing waves. Here's a picture of these standing waves, at their maximum amplitude, for $n = 1$, 3, 5, and 7 (and with $L = 3$).



Of the waves shown, the only one which satisfies $y_x(L/3,t) = 0$ is the $n = 3$ wave (in orange). You should be able to see that the wave $\sin\left(\frac{n\pi x}{2L}\right)$ has derivative zero at $x = L/3$ just when n is divisible by 3. So the general solution for this new flute is

$$u(x,t) = \sum_{n \text{ odd}\geq 1} C_{3n} \sin\left(\frac{3n\pi x}{2L}\right) \sin\left(\frac{3vn\pi t}{2L}\right). \tag{2}$$

The lowest angular frequency of the holeless flute is the coefficient of t in the $n = 1$ term, which is $v\pi/2L$. This measures the speed of oscillations in radians per second, while Hertz are cycles per second, so the fundamental frequency is

$$432\,\mathrm{Hz} = \frac{v\pi/2L}{2\pi} = \frac{v}{4L}.$$

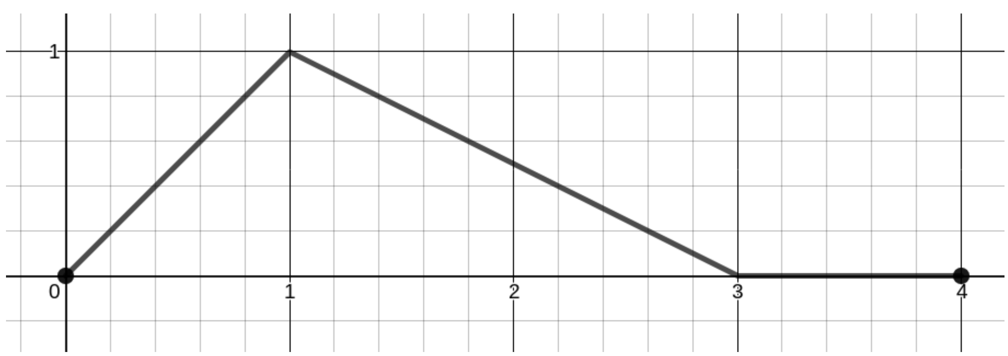
Likewise, the fundamental frequency of the new flute is

$$\frac{3v}{4L}.$$

This is three times the old fundamental frequency, so the answer is 1296 Hz.

(If you thought this was easy: what happens if you drill the hole at $x = 2L/3$ instead? And if you think that's easy: what about if you drill the hole at $x = L/\sqrt{3}$?)

3. *Here's the graph of the position of a string with fixed ends at $t = 0$:*

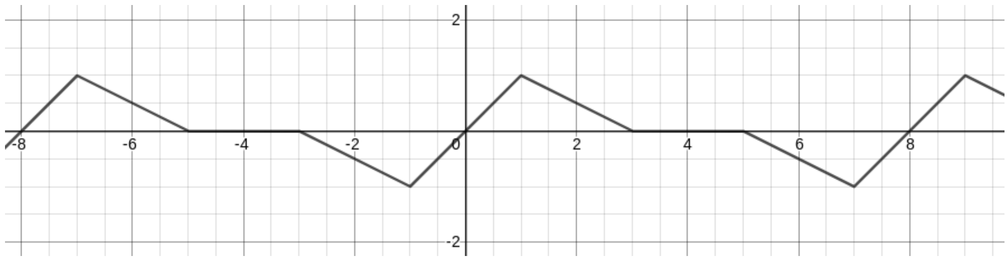


Sketch graphs of the position of the string at $t = 1$, $t = 2$, and $t = 3$. The length of the string is 4, the wave speed is 1, and the initial velocity is zero at each point.

One way to do this is to use Fourier series to get an explicit solution to the wave equation, $y(x,t)$, with the given initial conditions, and then plug in x and t. However, it's much easier to use the d'Alembert solution. Since the initial velocity is 0, this is

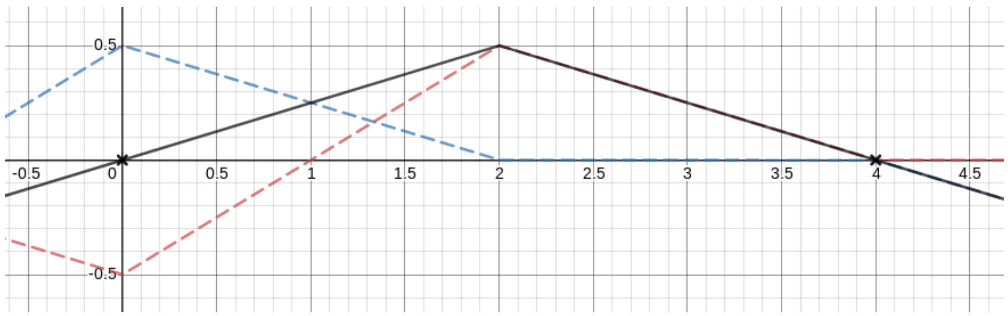
$$y(x,t) = \frac{1}{2}(F_{\text{odd}}(x+vt) + F_{\text{odd}}(x-vt)) = \frac{1}{2}(F_{\text{odd}}(x+t) + F_{\text{odd}}(x-t)),$$

where $F_{\text{odd}}(x)$ is the odd, $2L$-periodic (i.e. 8-periodic) extension of the initial position function $f(x)$. Here is F_{odd}:

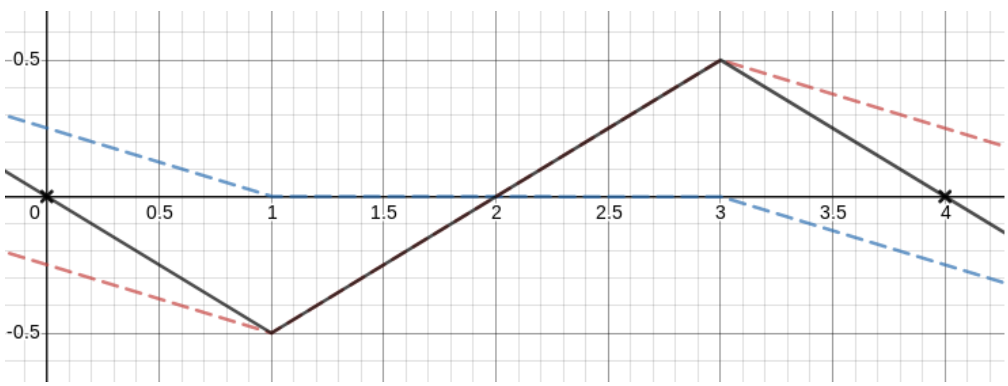


The maximum x-value of this function we really care about is $L + 3v = 7$, and the minimum is $0 - 3v = -3$.

Now, at $t = 1$, I've drawn $\frac{1}{2}F_{\text{odd}}(x+1)$ dotted in red, $\frac{1}{2}F_{\text{odd}}(x-1)$ dotted in blue, and their sum in black:



Here is $t = 2$:



Here is $t = 3$:

You can access an animated graph of this as `https://www.desmos.com/calculator/f1iyv64dzf`. What do you observe about the motion of the string? What happens when $t = 4$, so both of the travelling waves have moved by half their wavelength, that is, by a full length of the string? What, mathematically speaking, is keeping the ends of the string fixed?