Math 303, Homework 11 solutions

Suppose you have a metal washer of the form $\{(r,\theta) : 1 \le r \le 2\}$. *The inside circle is heated to a fixed temperature distribution* $f(\theta)$, *and the outside circle is heated to a fixed temperature distribution* $g(\theta)$.

(a) *Find the general formula for the steady-state temperature function* $u(r,\theta)$ *on the washer.*

The temperature distribution u satisfies the steady-state heat equation,

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \quad (a \le r \le b). \tag{1}$$

For the moment, we'll ignore the nonhomogeneous conditions on the boundary, but we will remember the periodicity condition

$$u(r, \theta + 2\pi) = u(r,\theta). \tag{2}$$

We solve the problem (1), (2) by separating variables. Initially, this goes identically with our solution of the steady-state heat equation on a disk in class. Let $u(r,\theta) = R(r)\Theta(\theta)$. Then the periodicity condition (2) tells us that Θ is 2π-periodic,

$$\Theta(\theta + 2\pi) = \Theta(\theta).$$

The differential equation (1) becomes

$$r^2 R''\Theta + rR'\Theta + R\Theta'' = 0.$$

We can change this into

$$-\frac{\Theta''}{\Theta} = \frac{r^2 R'' + rR'}{R} = \lambda,$$

where we observe that, since each side is independent of one of the two variables, they must both be equal to the same constant. The one-variable problem for Θ is then

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(\theta + 2\pi) = \Theta(\theta).$$

<u>$\lambda < 0$</u>: The general solution to the ODE is

$$\Theta(\theta) = C_1 e^{\sqrt{-\lambda}\theta} + C_2 e^{\sqrt{\lambda}\theta}.$$

But no such function (besides the zero function) is 2π-periodic.

$\underline{\lambda = 0}$: The general solution is

$$\Theta(\theta) = C_1 + C_2\theta.$$

This is periodic iff $C_2 = 0$, in which case it's a constant function. Let's write $\lambda_0 = 0$ and $\Theta_0(\theta) = 1$.

$\underline{\lambda > 0}$: The general solution is

$$\Theta(\theta) = C_1 \cos(\sqrt{\lambda}\theta) + C_2 \sin(\sqrt{\lambda}\theta).$$

This has period $2\pi/\sqrt{\lambda}$. Moreover, any integer multiple of $2\pi/\sqrt{\lambda}$ is also a period of this function. Since we want 2π to be a period, $\sqrt{\lambda}$ must be an integer. So we get eigenvalues

$$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots,$$

and associated eigenfunctions

$$\Theta_n = C_1 \cos(n\theta) + C_2 \sin(n\theta).$$

Notice that there's now a two-dimensional vector space of eigenfunctions for each eigenvalue. We could give names to a basis – for example.

$$\Theta_n^A = \cos(n\theta), \quad \Theta_n^B = \sin(n\theta).$$

Or we could just remember that the whole vector space needs to be taken into account.

We now return to the equation for R, which simplifies to

$$r^2 R'' + rR' - \lambda R = 0.$$

This is a bit tricky, but fortunately, we solved it in class, and you're welcome to just refer to the results we established there. For the sake of reference, I'll go over how to do it again. There are cases.

$\underline{\lambda = n^2 > 0:}$ So the equation is

$$r^2 R'' + rR' - n^2 R = 0.$$

We look for solutions of the form $R = r^a$. Such a solution satisfies

$$r^2 \cdot a(a-1)r^{a-2} + r \cdot ar^{a-1} - n^2 r^a = 0,$$

and since $r > 0$, we can divide by r^a to get

$$a(a-1) + a - n^2 = a^2 - n^2 = 0.$$

So $a = \pm n$, meaning that r^n and r^{-n} are solutions. Since the ODE is linear and homogeneous, anything of the form

$$R_n = C_1 r^n + C_2 r^{-n}$$

is a solution, and since it's second-order, these must be all the solutions. (In class, we discarded the solution r^{-n}, because it was not continuous at 0, but we can't do that here, as 0 is no longer part of the domain!)

<u>$\lambda = 0$:</u> The equation is

$$r^2 R_0'' + r R_0' = 0.$$

The same argument as above works, but it just gives us one solution, $R_0 = r^0 = 1$. We could use this to find a second linearly independent solution by the method of reduction of order. Alternatively, we can reduce the equation to first-order by writing $S(r) = R_0'(r)$, so we have

$$r^2 S' + rS = 0$$

or

$$\frac{dS}{dr} = -\frac{1}{r} S.$$

We can move things around and integrate this:

$$\begin{aligned}
\int \frac{1}{S}\, dS &= -\int \frac{1}{r}\, dr \\
\ln(S) &= -\ln(r) + C = \ln(1/r) + C \\
S &= \frac{C_1}{r} \\
R_0 &= \int S\, dr = C_1 \ln(r) + C_2.
\end{aligned}$$

So the second linearly independent solution in this case is $R(r) = \ln(r)$. Again, this wouldn't work as a solution on the disk, because it's discontinuous at 0, but it does work as a solution on the washer.

<u>Putting it all together:</u> For each λ_n, we have a two-dimensional space of possible Θ_n and a two-dimensional space of possible R_n. Any product of the form $R_n \Theta_n$ is a solution to the problem. For example, when $n = 1$, we have four linearly independent solutions,

$$r\cos(\theta), \quad r\sin(\theta), \quad r^{-1}\cos(\theta), \quad r^{-1}\sin(\theta).$$

Any linear combination of these is also a solution – anything of the form

$$A_1 r\cos(\theta) + B_1 r\sin(\theta) + C_1 r^{-1}\cos(\theta) + D_1 r^{-1}\sin(\theta).$$

Taking the sum over all n, we get a general solution

$$A_0 + B_0 \ln(r) + \sum_{n=1}^{\infty} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) + C_n r^{-n} \cos(n\theta) + D_n r^{-n} \sin(n\theta)\right). \tag{3}$$

Another way of writing this is

$$A_0 + B_0 \ln(r) + \sum_{n=1}^{\infty} \left[\left(E_n r^n + F_n r^{-n}\right)\left(G_n \cos(n\theta) + H_n \sin(n\theta)\right)\right].$$

In either case, there are four independent pieces of information that have to be specified for each n. This reduces to the previous line by defining $A_n = E_n G_n$, $B_n = F_n G_n$, and so on.

It doesn't matter what letters you use or how you arrange your solution, but your solution is wrong if you use the same notation for coefficients that should be different. For instance,

$$A + B \ln(r) + \sum_{n=1}^{\infty} \left(Cr^n \cos(n\theta) + Dr^n \sin(n\theta) + Er^{-n} \cos(n\theta) + Fr^{-n} \sin(n\theta)\right)$$

is an incorrect solution. This says that the coefficient of $r \cos(\theta)$ is the same as the coefficient of $r^2 \cos(2\theta)$, $r^3 \cos(3\theta)$, and so on, which is not right.

(b) *Find u when $f(\theta) = \cos(\theta)$ and $g(\theta) = \sin(\theta)$.* Notice that $f(\theta) = r(1, \theta)$ and $g(\theta) = r(2, \theta)$. Let's apply this to (3). We have

$$f(\theta) = r(1, \theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\theta) + B_n \sin(n\theta) + C_n \cos(n\theta) + D_n \sin(n\theta)\right),$$

$$\begin{aligned} g(\theta) = r(2, \theta) = A_0 + B_0 \ln(2) + \sum_{n=1}^{\infty} \big(&A_n 2^n \cos(n\theta) + B_n 2^n \sin(n\theta) \\ &+ C_n 2^{-n} \cos(n\theta) + D_n 2^{-n} \sin(n\theta)\big). \end{aligned}$$

No matter what f and g are, they're 2π-periodic functions and so have Fourier series. So the right-hand sides of these equations must be their Fourier series – in other words, $A_n + C_n$ is the coefficient of $\cos(n\theta)$ in the Fourier series for f, $B_n + D_n$ is the coefficient of $\sin(n\theta)$ in the Fourier series for f, and so on.

In this case, we have

$$\cos(\theta) = A_0 + \sum_{n=1}^{\infty} \left((A_n + C_n) \cos(n\theta) + (B_n + D_n) \sin(n\theta)\right),$$

$$\sin(\theta) = A_0 + B_0 \ln(2) + \sum_{n=1}^{\infty} \left((2^n A_n + 2^{-n} C_n) \cos(n\theta) + (2^n B_n + 2^{-n} D_n) \sin(n\theta)\right).$$

The Fourier series for $\cos(\theta)$ has a single nonzero term, $\cos(\theta)$. Comparing coefficients, we have

$$\begin{aligned} A_0 &= 0, & & \\ A_1 + C_1 &= 1, & B_1 + D_1 &= 0, \\ A_n + C_n &= 0, & B_n + D_n &= 0 \quad (n > 1). \end{aligned}$$

Likewise, comparing coefficients in the second equation of Fourier series gives

$$\begin{aligned} A_0 + B_0 \ln(2) &= 0, & & \\ 2A_1 + \frac{1}{2}C_1 &= 0, & 2B_1 + \frac{1}{2}D_1 &= 1, \\ 2^n A_n + 2^{-n} C_n &= 0, & 2^n B_n + 2^{-n} D_n &= 0 \quad (n > 1). \end{aligned}$$

We see that $A_0 = B_0 = 0$, and that $A_n = B_n = C_n = D_n = 0$ for each $n > 1$. The systems of equations for $n = 1$ are easily solved to show

$$A_1 = -1/3, \quad B_1 = 2/3, \quad C_1 = 4/3, \quad D_1 = -2/3.$$

So the solution is

$$u(r, \theta) = -\frac{1}{3} r \cos(\theta) + \frac{2}{3} r \sin(\theta) + \frac{4}{3} r^{-1} \cos(\theta) - \frac{2}{3} r \sin(\theta). \tag{4}$$

(c) *Find u when $f(\theta) = 1$ and $g(\theta) = 2$.* In this case, we have

$$1 = A_0 + \sum_{n=1}^{\infty} \left((A_n + C_n) \cos(n\theta) + (B_n + D_n) \sin(n\theta) \right),$$

$$2 = A_0 + B_0 \ln(2) + \sum_{n=1}^{\infty} \left((2^n A_n + 2^{-n} C_n) \cos(n\theta) + (2^n B_n + 2^{-n} D_n) \sin(n\theta) \right).$$

Comparing coefficients, we get

$$\begin{aligned} A_0 &= 1, & A_0 + B_0 \ln(2) &= 2, \\ A_n + C_n &= 0, & 2^n A_n + 2^{-n} C_n &= 0 \quad (n > 0), \\ B_n + D_n &= 0, & 2^n B_n + 2^{-n} D_n &= 0 \quad (n > 0). \end{aligned}$$

This implies that $A_0 = 1$, $B_0 = 1/\ln(2)$, and $A_n = B_n = C_n = D_n = 0$ for all other n. So the solution is

$$u(r, \theta) = 1 + \frac{\ln(r)}{\ln(2)} = 1 + \log_2(r). \tag{5}$$

Notice that this is independent of θ, which makes sense, because f and g are independent of θ, so the whole situation is symmetric under rotation.

It's interesting to me that this is a way of "physically" (though imprecisely and impractically) producing a logarithm function – you could literally heat a large washer in this way and measure the temperature at various values of r, to calculate the values of $\log(r)$.