Math 303, Homework 1 solutions

August 28, 2019

1. *Find a solution to the initial value problem:*

$$\begin{aligned} x_1' &= 3x_1 - 4x_2, & x_1(0) &= 2, \\ x_2' &= 4x_1 + 3x_2, & x_2(0) &= 2. \end{aligned}$$

Let $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$. We can write this system as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \mathbf{x}.$$

The characteristic polynomial of the matrix is

$$\begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 16 = \lambda^2 - 6\lambda + 25.$$

The roots of this polynomial are the eigenvalues. They are

$$\lambda = \frac{6 \pm \sqrt{36 - 4 \cdot 25}}{2} = 3 \pm 4i.$$

Let's take the eigenvalue $\lambda = 3 + 4i$. An eigenvector for this eigenvalue is a solution to the system of equations

$$\begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There are many solutions to this, but one possible one is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

This eigenvalue-eigenvalue pair gives us a complex-valued solution to the differential equation:

$$\mathbf{x} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(3+4i)t}.$$

(Since the matrix has real entries, the conjugate of this $\mathbf{x}$*, which is the function* $\overline{\mathbf{x}} = \binom{-i}{1} e^{(3-4i)t}$*, is also a solution to the differential equation, and the two solutions together form a basis of the space of complex-valued solutions.)*

Let's rewrite our $\mathbf{x}$ in a form where we can see its real and imaginary parts.

$$\mathbf{x} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{3t} \cdot e^{4it} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{3t}(\cos(4t) + i\sin(4t)) = \begin{pmatrix} -\sin(4t) + i\cos(4t) \\ \cos(4t) + i\sin(4t) \end{pmatrix} e^{3t}.$$

Then we define

$$\mathbf{x}_1 = \mathrm{Re}(\mathbf{x}) = \begin{pmatrix} -e^{3t}\sin(4t) \\ e^{3t}\cos(4t) \end{pmatrix},$$

$$\mathbf{x}_2 = \mathrm{Im}(\mathbf{x}) = \begin{pmatrix} e^{3t}\cos(4t) \\ e^{3t}\sin(4t) \end{pmatrix}.$$

These are a basis of the space of real-valued solutions. *(You might want to check for yourself that these solutions are linearly independent, and that, if we took the real and imaginary parts of* $\overline{\mathbf{x}}$*, the solutions we'd get are linearly dependent with* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*.)*

Thus, the solution to the initial value problem is of the form

$$\mathbf{x}_0 = C_1\mathbf{x}_1 + C_2\mathbf{x}_2.$$

Plugging in the initial values, we get

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Clearly, $C_1 = C_2 = 2$. The solution is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2e^{3t}\sin(4t) + 2e^{3t}\cos(4t) \\ 2e^{3t}\cos(4t) + 2e^{3t}\sin(4t). \end{pmatrix}}$$

(Even if you chose different eigenvectors originally, you should have gotten to this solution! Here's an unorthodox thing you could do: instead of finding the space of real-valued solutions at all, you could use the complex-valued basis $\mathbf{x}, \overline{\mathbf{x}}$*,*

and try to find complex numbers C_3 and C_4 such that $C_3\mathbf{x}+C_4\overline{\mathbf{x}}$ solves the IVP. This is entirely possible to do! If you then use Euler's formula to replace the complex exponentials with sines and cosines, you'll arrive at the same answer as the one above.)

2. *Find a system of the form $\mathbf{x}' = A\mathbf{x}$, where A is a constant coefficient matrix, such that two solutions are*

$$\mathbf{x}(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} 6 \\ 4 \end{pmatrix} e^{4t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.$$

The first thing you might want to notice is that the function

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \tag{1}$$

is also a solution to the differential equation. This is because the differential equation is linear and homogeneous (so its set of solutions forms a vector space), and this function is a linear combination of the two given. You don't have to do this, but it does clean things up a bit.

There are now two ways to go. One is to remember that, if we solve the equation $\mathbf{x}' = A\mathbf{x}$ using the eigenvalue method, we get solutions of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t},$$

where λ is an eigenvalue of A, and $\mathbf{v}$ is an associated eigenvector. So, to produce the solutions above, $\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)$ should be an eigenvector of A with eigenvalue 4, and $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ should be an eigenvector with eigenvalue -1. Since A is a 2×2 matrix, it only has at most two linearly independent eigenvectors. Then you could use this data to find the matrix.

The other, which is in some ways more straightforward *(and will work in problems like this where you* don't *have access to something like the eigenvalue method!)*, is to just plug in the two solutions to the differential equation and solve for the entries of A. That is, let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Plugging the first given solution into the differential equation gives

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot 4e^{4t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

We can cancel the e^{4t} (which is never zero) to get

$$\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \begin{pmatrix} 3a + 2b \\ 3c + 2d \end{pmatrix}.$$

Doing the same to the solution (1) gives

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a-b \\ c-d \end{pmatrix}.$$

(Again, if you didn't find (1)*, you could have gotten a second set of equations from the second solution given.)* Now we just solve for the unknown matrix entries. We end up with

$$\boxed{A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.}$$

3. *For each of the following scenarios, write a differential equation, or system of differential equations, describing the situation. Make sure to say explicitly what all your variables mean, any relevant initial values, and any important assumptions you make. Then, make an educated guess as to what will happen to the variables over time. You don't need to solve your equations.*

   A 10-kg mass hangs from a ceiling by a spring with spring constant 2 N/m. Another 10-kg mass hangs from the first mass by another spring with spring constant 1 N/m. Both springs are undamped.

   Let's first consider a single mass m, hanging from a single spring with spring constant k. *(You don't have to have done any of the next two paragraphs; the point is to convince you that you don't have to worry about gravity.)* When the system is at equilibrium, the weight mg of the mass is exactly balanced by the upward force of the spring. By Hooke's law, this force is $k\ell$, where ℓ is the length that the mass causes the spring to stretch (at equilibrium), beyond what its length (at equilibrium) would be if no mass were attached. That is,

   $$mg = k\ell.$$

   Now, if the system is not at equilibrium, the displacement x from equilibrium is a function of time t. I'll orient x so that positive x means an upward displacement, meaning the spring is compressed. I'm also defining it so that $x = 0$ is the equilibrium position of the mass, at which point the spring is already stretched a length ℓ. For an arbitrary value of x, the total stretch in the spring is then $\ell - x$. Using Newton's second law, we then have

   $$mx'' = (\text{force due to gravity}) + (\text{force due to spring}) = -mg + k(\ell - x).$$

   Let's check the signs. The gravitational force is down, so it has a minus sign. If the spring is stretched further, then $\ell - x$ is positive, and we have a positive (upwards) force for the spring, as expected.

Since $mg = k\ell$, this equation simplifies to

$$mx'' = -kx.$$

But this is exactly the same equation as if there were no gravity. The only difference is that the equilibrium position of the mass would be higher if there were no gravity. The upshot is that, *if we define $x = 0$ to be the equilibrium position of the mass*, we can ignore gravity in setting up the equations. This applies just as well to our two-spring system – and if you're not convinced, you should try doing the same sort of argument there for yourself!

Now we turn to actually solving the problem. Let x_1 and x_2 be the displacements of the two masses in meters, chosen so that $x_1 = x_2 = 0$ when the masses are at equilibrium, and so that the positive direction is upwards. Let spring A be the top spring (connected to mass 1 and the ceiling) and spring B the bottom spring (connected to mass 2 and mass 1). Using Newton's second law, and ignoring gravity as discussed above, we have two equations,

$$\begin{aligned} 10x_1'' &= (\text{force due to spring A}) + (\text{force due to spring B}), \\ 10x_2'' &= (\text{force due to spring B}). \end{aligned}$$

The stretch in spring A is $-x_1$. Thus, it exerts a force on mass 1 of $-2x_1$ Newtons in the upwards direction. The stretch in spring B is $x_1 - x_2$ – note this is a positive number if the spring is stretched and a negative number if it's compressed. So it exerts a force on mass 1 of $-(x_1 - x_2)$, and a force on spring 2 of $x_1 - x_2$. The equations we get are

$$\boxed{\begin{aligned} 10x_1'' &= -x_1 - (x_1 - x_2) = x_2 - 2x_1, \\ 10x_2'' &= x_1 - x_2. \end{aligned}}$$

Since the springs are undamped, if they're given any initial displacement at all they will oscillate forever, though how they do so depends on how they're displaced.

4. *A forest contains deer and wolves. If there were no wolves, the deer population would grow at a rate proportional to the current population. If there were no deer, the wolf population would decrease at a rate proportional to the current population. When the two animals live together, each wolf eats a constant amount of deer per year, and this increases the population of wolves at a rate proportional to both the deer population and the wolf population.*

Let D be the population of deer and W be the population of wolves. Each population's rate of change is affected by two separate factors: how that population interacts with itself (its natural birth and death rate) and how it interacts with the other population (in the deer's case, the wolves eat them; in the wolves' case, access to prey gives them more energy for reproduction.) Assuming these two interactions work independently, we have

$$\begin{aligned} D' &= \text{(natural birth/death rate)} + \text{(rate of being eaten by wolves)}, \\ W' &= \text{(natural birth/death rate)} + \text{(increased reproduction due to eating deer)}. \end{aligned}$$

We just have to figure out these four terms. We can observe the "natural birth/death" rates by observing each population in isolation. The problem tells us that

$$\begin{aligned} \underline{(W=0)} \quad D' &= \alpha D, \\ \underline{(D=0)} \quad W' &= -\beta W, \end{aligned}$$

where α and β are positive proportionality constants. Let γ be the amount of deer each wolf eats per year. Then the total amount of deer eaten per year is γW, so the second term in the expression for D' is $-\gamma W$. Finally, the increase in the wolf population due to predation is proportional to D and W, so of the form δDW, where δ is a fourth positive proportionality constant. The answer is

$$\boxed{\begin{aligned} D' &= \alpha D - \gamma W, \\ W' &= -\beta W + \delta DW. \end{aligned}}$$

All the constants are positive, and all are different (they're even in different units).

This is almost the standard predator-prey system but not quite: in that system the term $-\gamma W$ would be replaced by one of the form $-\gamma DW$. So if there are fewer deer around, the wolves eat fewer deer, which is perhaps more realistic.

It's impossible to predict what happens without assigning values to the constants. For example, if β is very large compared to δ, the wolves will die off, and then the deer will grow exponentially. If γ is very large compared to α, the wolves will eat all the deer and then die off. If the coefficients are balanced in the right way, it's possible for the two populations to oscillate, with more deer leading to more wolves to eat them, leading to fewer deer, leading to fewer wolves, et cetera.

5. *A house has a main floor, which is insulated from the cold outdoors, and an attic, which is not. Heat flows from the main floor to the attic, and from the attic to the outdoors, following Newton's law of cooling: the rate of temperature change of one space caused by its interaction with another space is proportional to the temperature difference. Initially, the temperature outside is* 40° F*, and the temperature in the attic and main floor is* 50° F*. There is also a heater on the main floor which increases its temperature by* 20° F *per hour.*

   Newton's law of cooling says that, if one object A is in contact with another object B, then the temperature of A changes according to the equation

   $$T_A'(t) = k(T_B - T_A).$$

   Here k is a positive constant that depends on the two substances. Note that if B is hotter than A, then T_A' is positive, as expected.

   Here, there are two variables worth worrying about, the temperature of the attic T_A and the temperature of the main floor T_M. *(The temperature of the outdoors won't be noticeably affected by the heat coming from one house, though you're welcome to include it; what this means is that the corresponding proportionality constant is really really small.)* There are three interactions: the interaction of the main floor with the attic, the interaction of the attic with the main floor, and the interaction of the attic with the outdoors. *(We're ignoring the interaction of the main floor with the outdoors because of the insulation. In reality, no insulation is perfect, but again, the proportionality constant that would appear would be quite small.)* Finally, there is also the heater, which contributes an additional 20° F/hr to T_M'. Combining all these effects and including the initial values, we get

   $$\boxed{\begin{aligned} T_A' &= k_{AM}(T_M - T_A) + k_{AO}(40 - T_A), && T_A(0) = 50, \\ T_M' &= k_{MA}(T_A - T_M) + 20, && T_M(0) = 50. \end{aligned}}$$

   Here, k_{AM}, k_{AO}, and k_{MA} are three different proportionality constants, the temperatures are in degrees Fahrenheit, and the differential equations are in degrees Fahrenheit per hour.

   What happens over time depends on the values of the constants. T_M cannot decrease much because of the heater, but it may level off if k_{MA} and k_{AO} are large (meaning that a lot of heat is lost to the outdoors). T_A will not decrease below 40, but may sink near there if k_{AO} is large, or rise considerably if k_{AM} is large compared to k_{AO}. If the heat loss rates are small, the heater will allow T_M to grow nearly linearly, which is not very realistic after even an hour or two!