Math 303, Homework 2 solutions

September 11, 2019

1. *Find the general solution to the system*

$$\mathbf{x}' = \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix}. \tag{1}$$

What happens to solutions to this system as $t \to \infty$?

This *should* have said

$$\mathbf{x}' = \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix} \mathbf{x}, \tag{2}$$

since this is the kind of system we've been talking about in class! Since I didn't notice the mistake, I'll accept correct solutions to (2) or to the equation I actually wrote.

Let's begin with the more interesting problem, (2). The characteristic polynomial of the matrix is

$$\begin{vmatrix} 14-\lambda & 9 \\ -25 & -16-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1.$$

Thus, there is a repeated eigenvalue of $\lambda = -1$. We now try to find eigenvectors with this eigenvalue. Such vectors satisfy

$$\begin{pmatrix} 15 & 9 \\ -25 & -15 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}.$$

One example is $\mathbf{x} = \left(\begin{smallmatrix} 3 \\ -5 \end{smallmatrix}\right)$, and any other example is a scalar multiple of this one. (If this were not true, then the matrix $A + I = \left(\begin{smallmatrix} 15 & 9 \\ -25 & -15 \end{smallmatrix}\right)$ would have a two-dimensional kernel, and the only 2×2 matrix with this property is the zero matrix.

Thus, the eigenvalue -1 is defective. From the eigenvector we just found, we have a solution

$$\mathbf{x}_1 = \begin{pmatrix} 3 \\ -5 \end{pmatrix} e^{-t}.$$

Let's look for another solution of the form

$$\mathbf{x}_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix} te^{-t} + \mathbf{w}e^{-t}.$$

We have

$$\mathbf{x}_2' = \begin{pmatrix} -3 \\ 5 \end{pmatrix} te^{-t} + \left(\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \mathbf{w} \right) e^{-t}, \tag{3}$$

and

$$A\mathbf{x}_2 = \begin{pmatrix} -3 \\ 5 \end{pmatrix} te^{-t} + A\mathbf{w}e^{-t}. \tag{4}$$

(The first term can be calculated quickly using the fact that $\left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right)$ is an eigenvector of A with eigenvalue -1.) Setting (3) and (4) equal to each other and rearranging, we get

$$(A + I)\mathbf{w} = \begin{pmatrix} 15 & 9 \\ -25 & -15 \end{pmatrix} \mathbf{w} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

One possible value for $\mathbf{w}$ is $\left(\begin{pmatrix} -1/5 \\ 0 \end{pmatrix} \right)$ (note there are many other solutions, though).

Finally, we have

$$\mathbf{x}_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix} te^{-t} + \begin{pmatrix} -1/5 \\ 0 \end{pmatrix} e^{-t}.$$

The general solution is

$$\boxed{\mathbf{x} = C_1 \begin{pmatrix} 3 \\ -5 \end{pmatrix} e^{-t} + C_2 \left(\begin{pmatrix} 3 \\ -5 \end{pmatrix} te^{-t} + \begin{pmatrix} -1/5 \\ 0 \end{pmatrix} e^{-t} \right).}$$

If you made a different choice of $\mathbf{w}$, you might want to check that you ended up describing the same *set* of solutions.

As $t \to \infty$, e^{-t} approaches zero, and it does so faster than t grows (so te^{-t} also approaches zero). Thus, $\boxed{\text{all solutions approach zero as } t \to \infty}$.

Now let's consider the problem (1). First, the only way this question even makes sense is if $\mathbf{x}$ is a 2×2 matrix function, rather than the $n \times 1$ vector functions we've traditionally been using. If we write

$$\mathbf{x} = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix},$$

then we see that the equation given separates into four single-variable ODEs, of the form

$$x_{11}' = 14$$

and so on. We can solve these by integrating. The general solution is

$$\boxed{\mathbf{x} = \begin{pmatrix} 14t + C_{11} & 9t + C_{12} \\ -25t + C_{21} & -16t + C_{22} \end{pmatrix}.}$$

(Note that there are four different constants of integration, which makes sense as this is really a four-dimensional system.)

$\boxed{\text{As } t \to \infty \text{, the top two entries of the matrix } \mathbf{x} \text{ go to } +\infty \text{ and the bottom two go to } -\infty.}$

2. (a) *Find a* 3×3 *matrix* A *such that* 0 *is the only eigenvalue of* A*, and the space of eigenvectors of 0 has dimension 1.* (Hint: upper triangular matrices are your friend!)

   (b) *Find the general solution to* $\mathbf{x}' = A\mathbf{x}$*.*

   (a) As the hint suggests, let's try to find an upper triangular matrix with these properties. Since the only eigenvalue is 0, it must be a triple eigenvalue, which means that the diagonal entries of the upper triangular matrix must all be 0. So we can write

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose that $\mathbf{v} = (x, y, z)^T$ is an eigenvector of A with eigenvalue 0. Then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay + bz \\ cz \\ 0 \end{pmatrix}. \tag{5}$$

Clearly, any vector of the form $(x, 0, 0)^T$ will work. These already form a 1-dimensional vector space. So for A to satisfy the condition of the problem, there can't be any eigenvectors with $y \neq 0$ or $z \neq 0$. In other words, for any pair $(y, z) \neq (0, 0)$, either $ay + bz \neq 0$ or $cz \neq 0$. Taking $(y, z) = (1, 0)$, we get $a \neq 0$, and taking $(y, z) = (b, -a)$, we get $c \neq 0$. So one solution would be any matrix of the form

$$\boxed{A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } a \neq 0 \text{ and } c \neq 0.}$$

That's a deductive way to approach the problem – we ended up finding all the upper triangular matrices that satisfy the conditions. Another way would be to write down a random upper triangular matrix with zeros on the diagonal, and calculate its space of eigenvectors. Chances are, this will give you a working answer on the first try, unless you happened to put too many zeros in the matrix.

Another strategy is to use the rank-nullity theorem from linear algebra. The kernel of A is exactly its space of eigenvectors, so is supposed to have dimension 1. Since A is a 3×3 matrix, this means its rank is supposed to be 2. The rank is the number of linearly independent columns, so, for example, you could have A be an upper triangular matrix as above where the columns $(a, 0, 0)$ and $(b, c, 0)$ are linearly independent. (This is actually equivalent to the above condition – do you see why?)

(b) For definiteness, I'm going to pick

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Your solution will depend on your choice of A. For this A, one eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding solution to the differential equation is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(that is, $\mathbf{x}_1$ is a constant function of t). We next look for a solution of the form

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \mathbf{v}_2.$$

Plugging this into the equation and remembering that $A\mathbf{v}_1 = \mathbf{0}$ gives

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\mathbf{v}_2.$$

One possible choice of $\mathbf{v}_2$ is $(0, 1, 0)^T$. Thus, we have

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Following the guidelines in the book for handling eigenvalues that are repeated more than once, we next look for a solutiono of the form

$$\mathbf{x}_3 = \frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} t^2 + \begin{pmatrix}0\\1\\0\end{pmatrix} t + \mathbf{v}_3.$$

(The reason for the factor of 1/2 becomes clear if you try to solve the problem without it. You might also want to notice that it makes it so that $\mathbf{x}_3' = \mathbf{x}_2$.) Plugging this into the equation gives

$$\begin{pmatrix}1\\0\\0\end{pmatrix} t + \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} t + A\mathbf{v_3}.$$

Thus, $A\mathbf{v_3} = \mathbf{v_2}$. We can pick $\mathbf{v_3} = (0,0,1)^T$, in which case

$$\mathbf{x}_3 = \frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} t^2 + \begin{pmatrix}0\\1\\0\end{pmatrix} t + \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

So the general solution, for this A, is

$$\boxed{\mathbf{x} = \begin{pmatrix}C_3/2\\0\\0\end{pmatrix} t^2 + \begin{pmatrix}C_2\\C_3\\0\end{pmatrix} t + \begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix}.}$$

3. *A 0.5-kg mass is attached to a spring with spring constant* 2.5 *N/m. The spring experiences friction, which acts as a force opposite and proportional to the velocity, with magnitude 2 N for every m/s of velocity. The spring is stretched 1 meter and then released.*

   (a) *Find a formula for the position of the mass as a function of time.*

   First, we have to set up the differential equation. Let x be the displacement of the mass from equilibrium, measured in meters. The mass experiences a spring force of $-kx = -2.5x$, where the negative sign makes sure this is directed opposite to the displacement. The friction force is $-\gamma x' = -2x'$, directed opposite to the velocity. All together, we have

$$0.5x'' = -2x' - 2.5x$$

   or

$$x'' = -4x' - 5x.$$

   (You might want to check that the units balance, where everything's in implicit SI units.)

As we did in class, I'll convert this to a first-order system – it's also possible to solve it just as you learned how to solve second-order ODEs with constant coefficients in your previous diff eqs class. Let's introduce the variable $v = x'$. Then

$$v' = x'' = -5x - 4v.$$

So we have the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}.$$

The coefficient matrix has characteristic polynomial

$$\begin{vmatrix} -\lambda & 1 \\ -5 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 5.$$

The eigenvalues are the complex conjugate pair

$$\lambda = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.$$

An eigenvector for $\lambda = -2 + i$ is a solution to

$$\begin{pmatrix} 2-i & 1 \\ -5 & -2-i \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One such vector is $(1, -2+i)^T$. This gives us the complex-valued solution

$$\begin{aligned}\begin{pmatrix} 1 \\ -2+i \end{pmatrix} e^{(-2+i)t} &= \begin{pmatrix} 1 \\ -2+i \end{pmatrix} e^{-2t}(\cos t + i \sin t) \\ &= \begin{pmatrix} \cos t + i \sin t \\ -2\cos t - \sin t + i(\cos t - 2\sin t) \end{pmatrix} e^{-2t}.\end{aligned}$$

Since the system of differential equations is homogeneous and linear with real coefficients, the real and imaginary parts of this solution are also solutions:

$$\begin{aligned}\mathbf{x_1} &= \begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} e^{-2t} \\ \mathbf{x_2} &= \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix} e^{-2t}.\end{aligned}$$

The general solution is thus

$$\mathbf{x} = \left(C_1 \begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix} \right) e^{-2t}.$$

We are also given initial conditions: $x(0) = 1$, $v(0) = 0$ (since the mass is released rather than pushed or pulled with some initial velocity). For the general solution above,

$$\mathbf{x}(0) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To get the required initial conditions, we must have $C_1 = 1$, $C_2 = 2$. Thus, the solution is

$$\mathbf{x} = \left(\begin{pmatrix} \cos t \\ -2\cos t - \sin t \end{pmatrix} + 2 \begin{pmatrix} \sin t \\ \cos t - 2\sin t \end{pmatrix} \right) e^{-2t}.$$

The formula for the position of the mass is

$$\boxed{x(t) = e^{-2t}(\cos t + 2\sin t).}$$

(This is the first coordinate of the vector-valued solution we obtained; the second coordinate describes the mass's velocity. It's fine to give the whole vector function as your answer, just as long as you know what it means!)

(b) *How much time does it take the mass to complete one oscillation (to pass the equilibrium point, bounce back, and return travelling in the same direction)?*

What I'm asking for is the period of the mass. Note that this is well defined even though the mass isn't moving in perfectly regular cycles (rather, damping is making the amplitude decrease over time). In fact, $x(t) = 0$ (the mass passes the equilibrium point) whenever

$$e^{-2t}(\cos t + 2\sin t) = 0 \text{ or } \tan t = -0.5.$$

The tangent function is periodic with period π, meaning that $\tan(t + \pi) = \tan t$. So the mass passes 0 every π seconds. But to complete a full oscillation, it must pass 0 twice. So $\boxed{\text{the period is } 2\pi \text{ seconds}}$.

This is kind of a subtle point, but: if we asked instead how long it takes the mass to go from maximum to the next maximum, or from minimum to the next minimum, we'd get a very slightly different answer. This is because the presence of e^{-2t} in the formula moves the maxima and minima slightly from those of the unadorned trig function. This is why I defined the period the way I did.

There are a few other ways to do this problem. One is to say that both the sine and cosine function are oscillating with angular frequency 1 (and thus period π), so the whole function $x(t)$ is oscillating with period π. A more elaborated version of the same argument – and this is a useful trick in general – is to write

$$\cos t + 2\sin t = R\cos(t - \delta).$$

You can do this with any linear combination of sines and cosines with the same angular frequency. The statement is equivalent to

$$\cos t + 2\sin t = R\cos(t)\cos(\delta) + R\sin(t)\sin(\delta).$$

For these to be equal for all t, the coefficients of $\sin(t)$ and $\cos(t)$ must be equal to each other, so we have

$$R\cos(\delta) = 1, \quad R\sin(\delta) = 2.$$

Thus,

$$R = \sqrt{R^2\cos^2\delta + R^2\sin^2\delta} = \sqrt{1^2 + 2^2} = \sqrt{5},$$

and

$$\tan(\delta) = 2.$$

Since $\sin(\delta) > 0$, we have $0 \leq \delta \leq \pi$, so δ actually is $\tan^{-1}(2) \approx 1.11$ radians. Finally, we obtain

$$x(t) = e^{-2t}(\sqrt{5}\cos(t - \tan^{-1} 2)),$$

which clearly has angular frequency 1 and thus period 2π.

(c) *By what fraction has the amplitude of the motion decreased in this time?*

The amplitude at time t is proportional to e^{-2t}. (The calculation just above shows that it's actually equal to $\sqrt{5}e^{-2t}$, but you don't have to know this.) So in time 2π, it is multiplied by $\boxed{e^{-4\pi}}$. This is about 3×10^{-6}, which is a very rapid decrease.

(d) *Do the answers to (b) and (c) depend on the initial position of the mass? Why or why not?*

Any solution to the equation is of the form

$$x(t) = e^{-2t}(C_1\cos(t) + C_2\sin(t)).$$

The value of 2 appearing in the exponent which we used to solve (c), and the angular frequency of 1 which we used to solve (b), don't depend on the initial position (or velocity). So $\boxed{\text{the answers don't depend on the initial position}}$.

(e) *By immersing the spring in one of a variety of rare, delicious syrups, it's possible to increase the damping constant while keeping the spring constant the same. Can you increase the damping constant so that the spring doesn't oscillate at all, but just returns to its starting point? What's the smallest value of the damping constant that will do this?*

Suppose that the damping constant is γ. Then the second-order equation is instead

$$x'' = -2\gamma x' - 5x,$$

and the first-order system is instead

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -2\gamma \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}.$$

The characteristic polynomial of the coefficient matrix is

$$\lambda^2 + 2\gamma\lambda + 5,$$

which has roots

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - 5}.$$

Now we have to think about what the eigenvalues mean. If they are complex conjugates ($\gamma^2 - 5 < 0$), then the solutions will always be sums of sines and cosines, multiplied by an exponential function. This means that the spring will oscillate. If both solutions are real, on the other hand ($\gamma^2 - 5 > 0$), then the solutions are sums of exponential functions – the value of x just decays to 0, which is what we want. The critical value is exactly $\boxed{\gamma = \sqrt{5} \text{ N/(m/s)}}$. Note that this gives the coefficient matrix a critical eigenvalue, which means that the solution will probably have a term of the form $te^{-\gamma t}$ – however, this also behaves like exponential decay and does not lead to oscillations. This kind of behavior is called **critical damping**, and is desirable in mechanical applications. For example, a good hydraulic brake on a door is designed so that the door closes as quickly as possible, but has velocity near zero when it reaches the closed position (so it doesn't slam).