Math 303, Homework 4

Due September 19, 2019

Remember to explain your reasoning and show your work! You can use computer tools as long as you explain what you're doing.

1. The point of this problem is to get you to think more about how to use eigenvectors and eigenvalues in higher dimensions. Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   (a) Find the general solution to the system.

   (b) Some of the solutions travel in closed orbits in a single plane through the origin. What is the plane? *(There are multiple ways to specify a plane: you can give an equation of the form $ax + by + cz = 0$, or a pair of basis vectors.)*

   (c) Some of the solutions travel along a line through the origin. What is the line?

   (d) What do the answers to (b) and (c) have to do with the eigenvectors of the coefficient matrix? *(Hint: if you have a complex eigenvector, you may find it helpful to separate it into real and imaginary parts.)*

   (e) How would you describe a typical solution to the system (i.e., not one of the ones mentioned in (b) and (c))?

2. Consider the system

$$x' = \sin(y),$$
$$y' = x^2.$$

   (a) Find the critical points of the system and the Jacobian at each critical point.

   (b) Pick your favorite critical point and solve the linearization of the system at that point. How would you describe the behavior of solutions to the linearization?

(c) This system is *not* almost linear: although the functions involved are continuously differentiable, and although the system itself has isolated critical points, the linearizations do not have isolated critical points. Thus, the linearization is not a good approximation to the critical behavior of the system. How do solutions to the nonlinear system actually behave near the critical points?

3. Consider the system

$$x' = \sin(y),$$
$$y' = x + x^2.$$

(a) Find the critical points of the system and the Jacobian at each critical point.

(b) Pick two critical points with different x-values and find the eigenvalues of the Jacobian at those points. What kind of critical behavior does the linearization have at each point?

(c) Is this nonlinear system almost linear? How do its solutions behave near the critical points?