Math 303, Homework 4 solutions

1. *The point of this problem is to get you to think more about how to use eigenvectors and eigenvalues in higher dimensions. Consider the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   (a) *Find the general solution to the system.*

   I used WolframAlpha to find the eigenvalues and eigenvectors:

$$\begin{aligned} \lambda_1 &= 2, & \mathbf{v}_1 &= (0,0,1)^T, \\ \lambda_2 &= i, & \mathbf{v}_2 &= (1+i, i, 1)^T, \\ \lambda_3 &= -i, & \mathbf{v}_3 &= (1-i, -i, 1)^T. \end{aligned}$$

   The first pair gives the solution

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

   The second pair gives the complex-valued solution

$$\begin{aligned} \mathbf{y} = \begin{pmatrix} 1+i \\ i \\ 1 \end{pmatrix} e^{it} &= \begin{pmatrix} 1+i \\ i \\ 1 \end{pmatrix} (\cos t + i \sin t) \\ &= \begin{pmatrix} \cos t - \sin t + i\cos t + i \sin t \\ -\sin t + i \cos t \\ \cos t + i \sin t \end{pmatrix}. \end{aligned}$$

   As the coefficient matrix is real-valued, the real and imaginary parts of $\mathbf{y}$ are also solutions:

$$\mathbf{x}_2 = \begin{pmatrix} \cos t - \sin t \\ -\sin t \\ \cos t \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} \cos t + \sin t \\ \cos t \\ \sin t \end{pmatrix}.$$

Thus, the general real-valued solution is

$$\boxed{\mathbf{x} = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} \cos t - \sin t \\ -\sin t \\ \cos t \end{pmatrix} + C_3 \begin{pmatrix} \cos t + \sin t \\ \cos t \\ \sin t \end{pmatrix}.}$$

(b) *Some of the solutions travel in closed orbits in a single plane through the origin. What is the plane?*

We first need to figure out which solutions travel in closed orbits. Graphs can help here, as can thinking about what the various terms in the general solution are doing. The first term gives exponential growth in the z direction. The other two terms are made up of cosines and sines, and so both oscillate with period 2π. Thus, any solution with $C_1 = 0$ will oscillate with period 2π. These are the solutions we expect to stay in a plane.

As an example, let $C_2 = 1$ and $C_3 = 0$. Then the solution has the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t - \sin t \\ -\sin t \\ \cos t \end{pmatrix}.$$

This is an ellipse around the origin. Putting $t = 0$, we see that the plane should contain $(1, 0, 1)$. Putting $t = \pi/2$, we see that it also contains $(-1, -1, 0)$. As the plane goes through the origin, it must contain the vectors from the origin to these two points.

Thus, $\boxed{\text{the plane is spanned by the vectors } (1,0,1)^T \text{ and } (-1,-1,0)^T.}$

We can find an equation for it as follows: the cross product of the spanning vectors is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

This is a normal vector to the plane. So the equation of the plane is

$$\boxed{x - y - z = 0.}$$

It's not too hard to check that all solutions with $C_1 = 0$ lie in this plane: given such a solution, at time t it goes through the point

$$(C_2(\cos t - \sin t) + C_3(\cos t + \sin t), -C_2 \sin t + C_3 \cos t, C_2 \cos t + C_3 \sin t).$$

One can plug these values into the equation $x - y - z = 0$ to see that the point always satisfies this equation, no matter what t or the constants are.

The point of this is that, even in higher dimensions, a pair of conjugate imaginary eigenvalues will give solutions that orbit elliptically in some plane. Likewise, a pair of conjugate complex eigenvalues with nonzero real part will give spiral solutions in some plane. (Things get more complicated if the eigenvalues are also repeated, but I'll leave this as a thing for you to think about).

(c) *Some of the solutions travel along a line through the origin. What is the line?*

This is much easier. We know that any real eigenvalue λ gives rise to solutions $\mathbf{v}e^{\lambda t}$ that travel along a line parallel to $\mathbf{v}$. In this case, the only real eigenvalue we have is 2, and the corresponding eigenvector is $(0, 0, 1)$. Thus, there are solutions that travel along the $\boxed{z \text{ axis}}$, also known as the line $\boxed{x = y = 0}$, or the line

$$\boxed{\left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}.}$$

(d) *What do the answers to (b) and (c) have to do with the eigenvectors of the coefficient matrix?*

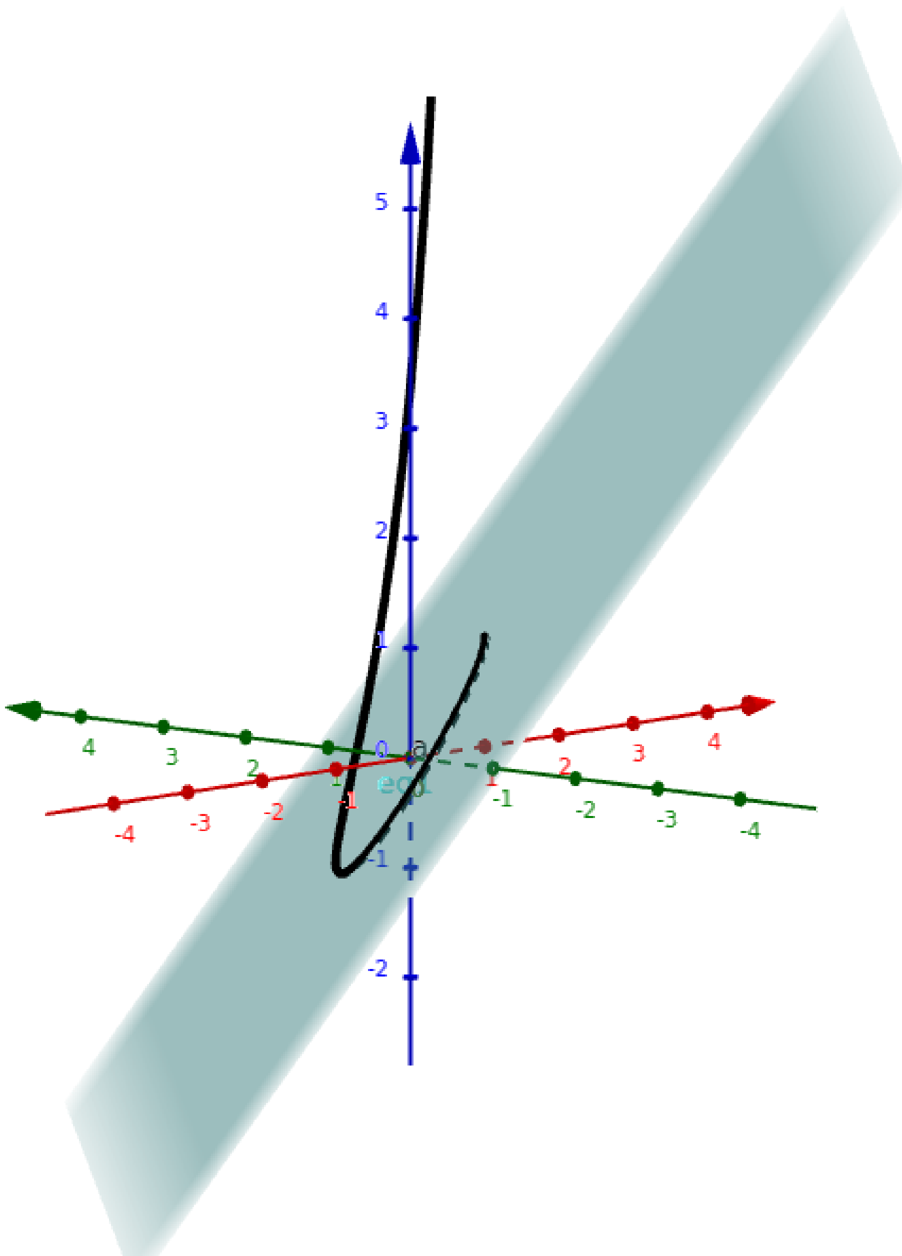
In (c), solutions travel linearly along the line parallel to the eigenvectors with real eigenvalue.

Analogously, we should maybe think that the other eigenvectors "point in the direction" of the plane containing the elliptical orbits. This is an imprecise statement, since these eigenvectors are not real. However, the real part of the nonreal eigenvector $\mathbf{v}_2$ is $(1, 0, 1)^T$, and its imaginary part is $\pm(1, 1, 0)^T$ – and these vectors span the plane! In fact, one could consider the eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$ themselves to be in a complexified version of the plane, in the sense that their coordinates satisfy the relation $x - y - z = 0$.

So: a real eigenvalue gives rise to solutions that grow or decay exponentially on a line, and that line is parallel to the associated eigenvectors. A pair of complex conjugate eigenvalues gives rise to solutions that orbit or spiral on a plane, and that plane is spanned by the real and imaginary parts of the eigenvectors to either eigenvalue.

(e) *How would you describe a typical solution to the system (i.e., not one of the ones mentioned in (b) and (c))?*

If neither C_1, nor both C_2 and C_3, is zero, then the solution will neither stay on a line nor orbit in a plane. Rather, it will have some characteristics of both. It will form a sort of corkscrew shape, orbiting parallel to the plane $x - y - z = 0$ with period 2π while simultaneously moving exponentially in the z direction. Here's a picture of one such solution with parameters $C_1 = 0.0002$, $C_2 = 1$, $C_3 = 0$. I've also graphed the plane $x - y - z = 0$, which the solution stays close to for small values of t.


5
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eq1

2. *Consider the system*

$$x' = \sin(y),$$
$$y' = x^2.$$

(a) *Find the critical points of the system and the Jacobian at each critical point.*

The critical points are the points with $x' = y' = 0$. If $y' = 0$, then $x^2 = 0$, so $x = 0$. If $x' = 0$, then $\sin(y) = 0$, so $y = k\pi$ where k is an integer $(\dots, -2, -1, 0, 1, 2, \dots)$. So $\boxed{\text{there is a critical point at } (0, k\pi)}$ for any integer k.

The Jacobian of the system is

$$J(x, y) = \begin{pmatrix} 0 & \cos(y) \\ 2x & 0 \end{pmatrix}.$$

At the critical point $(0, k\pi)$, we have

$$\boxed{J(0, k\pi) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & k \text{ even} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & k \text{ odd.} \end{cases}}$$

(b) *Pick your favorite critical point and solve the linearization of the system at that point. How would you describe the behavior of solutions to the linearization?*

At the critical point $(0, 0)$, the linearization of the system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(This is just the linear system whose coefficient matrix is the Jacobian at that critical point.)

This system has a repeated eigenvalue of 0. One eigenvector with this eigenvalue is $(1, 0)^T$. There is only a one-dimensional space of eigenvectors, so the eigenvalue is defective. Associated to this eigenvector is the solution

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We then look for a generalized eigenvector $\mathbf{w}$ with

$$(A - 0I)\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

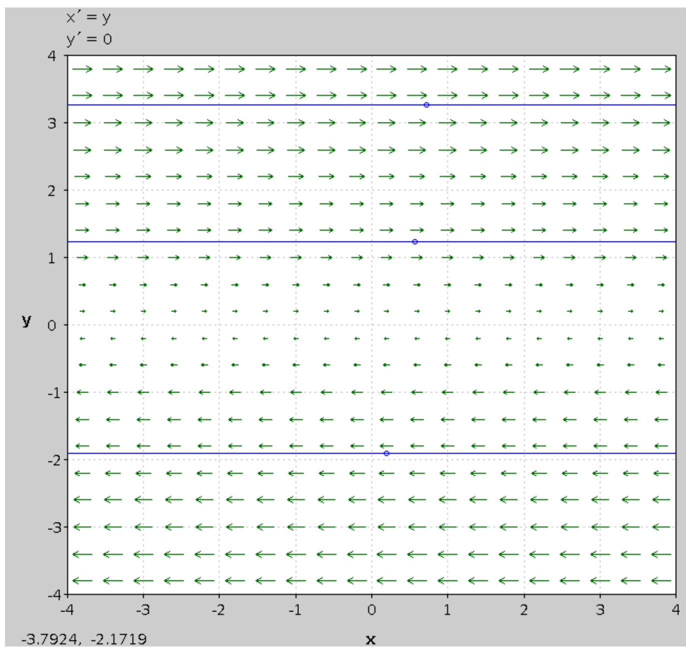
One such vector is $(0,1)^T$. Thus, there is a second solution

$$\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{0t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{0t} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t.$$

The general solution is then

$$\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right).$$

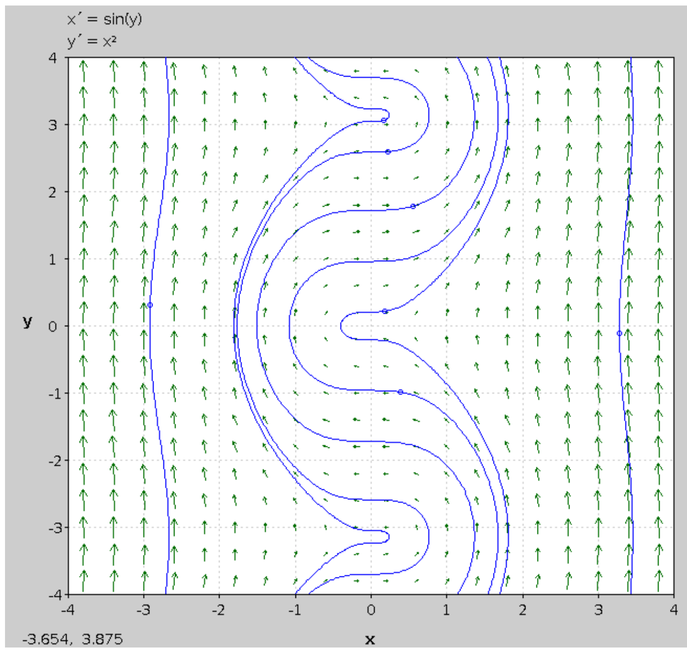
Any solution with $C_2 = 0$ is constant at some point on the x-axis. Thus, any point on the x-axis is a critical point of this linear system (though not of the nonlinear system in (a)!). At points not on the x-axis (with $C_2 \neq 0$), solutions travel linearly in the positive or negative x-direction. Here's a picture from `pplane`.



(c) *This system is not almost linear: although the functions involved are continuously differentiable, and although the system itself has isolated critical points, the linearizations do not have isolated critical points. Thus, the linearization is not a good approximation to the critical behavior of the system. How do solutions to the nonlinear system actually behave near the critical points?*

To explain what I wrote in the problem: even though they're infinitely many critical points, they're isolated because each one is a nonzero distance from all the others. That is, if you zoom in close to one of the critical points, you won't see any others. However, as we saw above, the linearization at $(0,0)$ has critical points at every point along the x-axis. Thus, the linearization does not have isolated critical points. This happens any time 0 is an eigenvalue of the Jacobian, and, in particular, happens at all the other critical points of our system.

Since we can't use the techniques we learned in class to answer this question, our only resort is to graph it.



This picture shows the critical points at $(0,0)$, $(0,\pi)$, and $(0,-\pi)$. As you can see, solutions that approach these critical points bend around them and travel in the opposite direction. Some solutions just slalom back and forth between critical points forever. This isn't a sort of behavior we've studied much, and these critical points aren't nodes, spiral points, or any of the other named ones we've seen.

3. *Consider the system*

$$x' = \sin(y),$$
$$y' = x + x^2.$$

(a) *Find the critical points of the system and the Jacobian at each critical point.*

The quadratic $x+x^2$ is zero just when $x = 1$ or -1. By the same reasoning as in 2(a), the critical points are at $\boxed{(0,k\pi) \text{ and } (-1,k\pi),}$ for every integer k.

(b) *Pick two critical points with different x-values and find the eigenvalues of the Jacobian at those points. What kind of critical behavior does the linearization have at each point?*

The Jacobian at an arbitrary point is

$$J(x, y) = \begin{pmatrix} 0 & \cos(y) \\ 1 + 2x & 0 \end{pmatrix}.$$

At $(0, 0)$, this is

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which has eigenvalues ± 1. The linearization at this point is a saddle point.

At $(-1, 0)$, the Jacobian is

$$J(-1, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which has eigenvalues $\pm i$. The linearization here is a center.

Note that 0 is not an eigenvalue of either of these matrices. The same is true at all the other critical points.

(c) *Is this nonlinear system almost linear? How do its solutions behave near the critical points?*

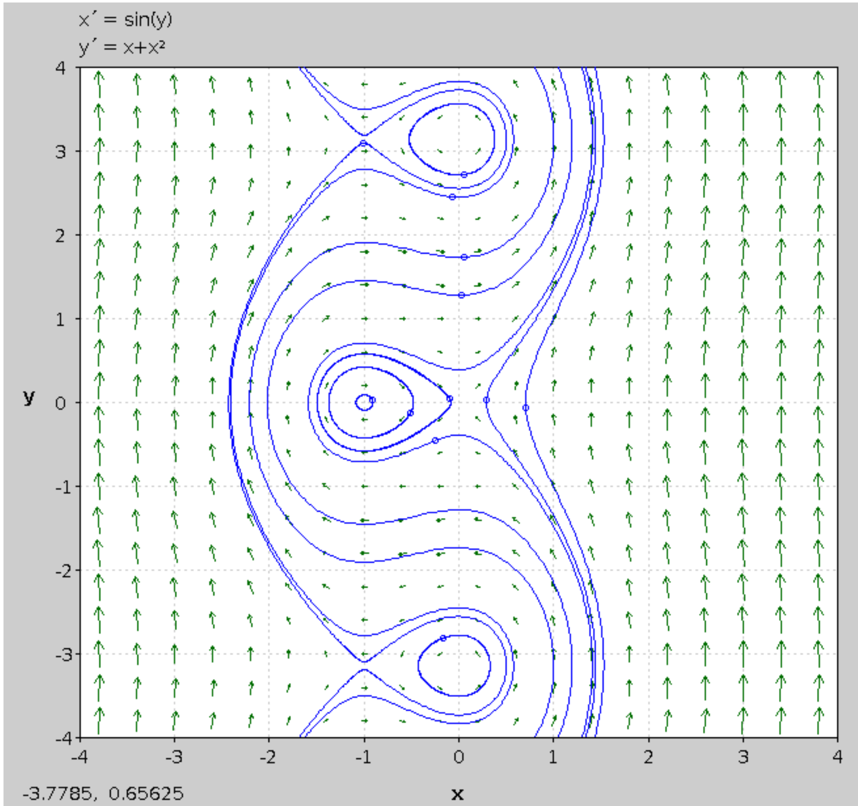
The system is almost linear because:

(1) the functions $\sin(y)$ and $x + x^2$ are continuously differentiable;

(2) the system has isolated critical points;

(3) the Jacobian does not have 0 as an eigenvalue at any critical point (equivalently, the linearization at any critical point has isolated critical points).

From part (b), we can see that the origin of the nonlinear system is a saddle point, and that the critical point $(-1, 0)$ is a center or possibly a spiral point (this is one of the edge cases we talked about). It's not too hard to do the same reasoning at all the critical points, and see that $(0, 2k\pi)$ and $(-1, (2k+1)\pi)$ are always saddle points, and that $(0, (2k+1)\pi)$ and $(-1, 2k\pi)$ are centers/spiral points. To see this in action, and confirm that the questionable points are in fact centers, we turn to `pplane`.


x′ = sin(y)
y′ = x+x²
y
x
-3.7785, 0.65625