Math 303, Homework 5 solutions

1. *Consider a competing-species system of the form*

$$x' = a_1x - b_1x^2 - c_1xy,$$
$$y' = a_2y - b_2y^2 - c_2xy,$$

where all the constants are positive.

Qualitatively speaking, there are a few different types of behavior such a system can have. For example, one possibility is that the species converge on a stable equilibrium at which both have nonzero population. Give a classification of the possible behaviors of this system, and say which values of the constants lead to which behavior.

I'll make three remarks first.

Remark 1. The ideal thing to do is to find all the critical points and their stabilities and make an assessment based on that. However, the Jacobian at the critical point where neither population is zero, is, I'm told, pretty nasty. Following the advice in my email, we're going to only consider the critical points where one of the population is zero, and we're going to assume that one of four things can happen: x always wipes out y, y always wipes out x, either x wipes out y or y wipes out x depending on the initial conditions, or the populations coexist peacefully forever. How do we know, for example, that we can't have a situation where x sometimes wipes out y and sometimes the populations coexist, depending on the initial conditions? The answer is that we don't, at first. We could only guess this by thinking carefully about the critical points and looking at lots of phase planes in `pplane`. Doing this kind of generalization from observations was one of my goals for you with this problem.

Remark 2. The book gives what I'd call an unjustified discussion of this question, saying that if $b_1b_2 > c_1c_2$ then "[self-]inhibition outweighs competition" and the non-axis critical point is stable, and that if $b_1b_2 < c_1c_2$ then it's unstable and one species wipes the other out. This is a great way of thinking about it, and I'm glad if you were able to get help from the book here. However, insights like this one have to be earned. Why do b_1b_2 and c_1c_2 have anything to do with anything? How do we know that the populations can ever peacefully coexist?

Why don't a_1 and a_2 appear in these inequalities? Again, we don't know the answers to any of these questions starting out. We can only get to them using mathematical reasoning.

Remark 3. The sort of answers we will get are inequalities: if $\alpha < \beta$ then something happens; if $\alpha > \beta$ then something else happens. But what if $\alpha = \beta$? In this problem, setting things exactly equal to each other tends to result in Jacobians with zero as an eigenvalue or other strange situations. These are degenerate cases of the problem, and if you asked me about them, I told you to ignore them. You're justified in doing so because any practical application of the competing-species model contains significant error, from the fact that the constants $a_1, \ldots, c_2$ must be estimated from the actual populations, and from the fact that we're using continuous variables x and y to model discrete populations. Knowing that two parameters are exactly equal or that an eigenvalue is exactly zero requires an infinite degree of precision, which we don't have. I've avoided all discussion of degenerate cases here, and have marked points where I've done so with a little circle$^\circ$. It's okay if you thought about these cases, though, and I'd be interested to here your conclusions!

All right, let's start solving the problem. The critical points are the points where

$$\begin{aligned} 0 &= a_1 x - b_1 x^2 - c_1 xy = x(a_1 - b_1 x - c_1 y), \\ 0 &= a_2 y - b_2 y^2 - c_2 xy = y(a_2 - b_2 y - c_2 x). \end{aligned}$$

Each equation is the product of two linear relations, so there are always$^\circ$ exactly four critical points, each corresponding to the intersection of one line from the first set with one line from the second set. The three easy ones are

$$(0,0), \quad (a_1/b_1, 0), \quad (0, a_2/b_2).$$

The Jacobian is
$$\begin{pmatrix} a_1 - 2b_1 x - c_1 y & -c_1 x \\ -c_2 y & a_2 - 2b_2 y - c_2 x \end{pmatrix}.$$

The critical point $(0,0)$: Here the Jacobian is

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Since both a_1 and a_2 are positive, the origin is an unstable node. We can even see that the tangent directions are the axes, corresponding to the situation where one species or the other is extinct, and that a_1 and a_2 describe the growth rate of the two populations near the origin, as one would expect.

The critical point $(a_1/b_1, 0)$: Here the Jacobian is

$$\begin{pmatrix} -a_1 & -c_1 a_1/b_1 \\ 0 & a_2 - c_2 a_1/b_1 \end{pmatrix}.$$

This is upper triangular, so its eigenvalues are $-a_1$ and $a_2 - c_2 a_1/b_1$. The eigenvalue $-a_1$ is always negative. The sign of the other eigenvalue depends on whether $a_2 > c_2 a_1/b_1$, or equivalently (since b_1 is positive) whether $a_2 b_1 > a_1 c_2^{\circ}$. We see that

if $a_2 b_1 > c_2 a_1$, the critical point is an unstable saddle point;
if $a_2 b_1 < c_2 a_1$, the critical point is an asymptotically stable node.

The critical point $(0, a_2/b_2)$: This is very similar to the previous critical point. The conclusion° is that

if $a_1 b_2 > c_1 a_2$, the critical point is an unstable saddle point;
if $a_1 b_2 < c_1 a_2$, the critical point is an asymptotically stable node.

What can we conclude so far? If a critical point is an asymptotically stable node, then at least within some small radius, nearby solutions converge to it. So, if one of the two axis critical points above is asymptotically stable, that means that it's *possible* for the species which is zero at that point to go extinct.

On the other hand, if a critical point is a saddle point, then solutions generally don't converge to it at all. In fact, there'll be just a single direction, corresponding to the direction of the eigenvectors of the Jacobian with negative eigenvalue, along which solutions can converge to this point. In this case, we can figure out what direction that is, either by using `pplane`, by finding the eigenvectors of the Jacobian, or with logic. For example, if $(a_1/b_1, 0)$ is a saddle point, then its negative eigenvalue is $-a_1$, and an eigenvector with this eigenvalue is $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Thus, solutions can converge to the critical point only along the x-axis. This is just the situation where the y population is *already* extinct, and the x population approaches its carrying capacity along the x axis. The conclusion is that, if the initial y population is nonzero, then it can *never* become zero. (Note that it can't approach the origin, either, because the origin is always unstable.)

So we can make the following conclusions (which are roughly what I expected from you, though a different analysis is fine if it's carefully argued).

I. If $a_2 b_1 > c_2 a_1$ and $a_1 b_2 > c_1 a_2$, then both axis critical points are unstable, so neither population can go extinct. Given the list of possibilities I gave you, this means that the populations have to coexist peacefully.

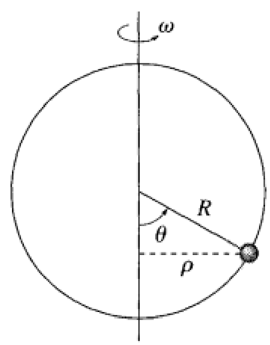
II. If $a_2 b_1 > c_2 a_1$ and $a_1 b_2 < c_1 a_2$, then the x-axis critical point is unstable but the y-axis critical point is stable. In this case, the ys can't go extinct but the xs can. This means (again, given the list of possibilities) that the ys must wipe out the xs.

III. If $a_2 b_1 < c_2 a_1$ and $a_1 b_2 > c_1 a_2$, then the situation is reversed and the xs must wipe out the ys.

IV. If $a_2 b_1 < c_2 a_1$ and $a_1 b_2 < c_1 a_2$, then both axis critical points are stable, so both populations can go extinct. Which one goes extinct must depend on the initial conditions.

Remark 4. By manipulating the inequalities above, you can conclude that $b_1 b_2 > c_1 c_2$ in case I, and $b_1 b_2 < c_1 c_2$ in case IV. In cases II and III, either thing could happen. This lets you reproduce part of the book's analysis, although I think the one above is more detailed than theirs.

Remark 5. Another thing you could think about is the location of the non-axis critical point. In cases II and III, this point is actually not in the first quadrant, so it doesn't correspond to an actual population of the two species. You can see it on `pplane` if you zoom out. In cases I and IV, it's in the first quadrant, and stable in case I, unstable in case IV.



2. *A hula hoop of radius R is stood on its end and rotated around the vertical axis with constant angular frequency ω. Inside the hula hoop is a marble of mass m, which can only move in the circle of the hula hoop. Writing θ for the marble's angle up from the vertical axis (so that $\theta = 0$ is the bottom of the hula hoop), the equation of motion for the marble is*

$$\theta'' = \left(\omega^2 \cos(\theta) - \frac{g}{R}\right) \sin(\theta). \tag{1}$$

You don't have to derive this equation, but the idea is that, from the point of view of the hoop, the marble experiences both gravity and a centrifugal "force" coming from the rotation of the hoop that pushes it outwards from the axis of rotation.

(a) *Find all the critical points of the system, determine the type and stability of each, and give a physical interpretation of your results. Your answer may depend on the values of the parameters.*

First, we need to change this second-order equation into a two-dimensional system. Let $x = \theta$ and $y = \theta'$. Then we can rewrite (1) as the system

$$x' = y,$$
$$y' = \left(\omega^2 \cos(x) - \frac{g}{R}\right) \sin(x).$$

The critical points are where $x' = y' = 0$, or equivalently where $\theta' = \theta'' = 0$. These are the points at which the marble is neither moving nor accelerating, which agrees with our intuition about physical equilibria.

If $x' = 0$, then $y = 0$. If $y' = 0$, then either $\sin(x) = 0$ (so $x = k\pi$ for some integer k), or $\cos(x) = \frac{g}{R\omega^2}$.

There are really only two cases where $\sin(x) = 0$: $x = 0$, the bottom of the hoop, and $x = \pi$, the top of the hoop. The other values of $k\pi$ physically mean the same thing.

What about when $\cos(x) = \frac{g}{R\omega^2}$? It isn't enough to write down $x = \cos^{-1}(g/R\omega^2)$ – we have to think about what this means. The arccosine function is defined with domain $[-1, 1]$ and range $[0, \pi]$. So, first, this expression only makes sense if $g/R\omega^2 \leq 1$. If $g/R\omega^2 > 1$, then we want $\cos(x)$ to be a number larger than 1, which is impossible. Second, if $g/R\omega^2 < 1$, then there are *two* meaningful angles with that cosine, namely $\pm \cos^{-1}(g/R\omega^2)$. (Third, there appears to be a degenerate case where $g/R\omega^2 = 1$, so these two critical points coincide with the one at the top of the hoop.)

So the critical points (up to shifts of x by multiples of 2π) are $(0, 0)$, $(\pi, 0)$, and, if $g \leq R\omega^2$, $(\pm \cos^{-1}(g/R\omega^2), 0)$.

The Jacobian of the system is

$$\begin{pmatrix} 0 & 1 \\ \omega^2(\cos^2(x) - \sin^2(x)) - \frac{g}{R}\cos(x) & 0 \end{pmatrix}.$$

The critical point $(0, 0)$: Here the Jacobian is

$$\begin{pmatrix} 0 & 1 \\ \omega^2 - \frac{g}{R} & 0 \end{pmatrix}.$$

The eigenvalues are

$$\pm\sqrt{\omega^2 - g/R}.$$

If $\omega^2 > g/R$, we have a saddle point. If $\omega^2 < g/R$, we have a center. If $\omega^2 = g/R$, the eigenvalues are zero, so this is a degenerate case.

This point corresponds to the bottom of the hoop. If the hoop is rotating slowly, this is a stable critical point: a marble close to the bottom of the hoop will roll back and forth across it. However, if the hoop is rotating quickly, this critical point is unstable. This is because the centrifugal force is strong enough to push the marble away as soon as it leaves the bottom of the hoop.

The critical point $(\pi, 0)$**:** Here the Jacobian is

$$\begin{pmatrix} 0 & 1 \\ \omega^2 + \frac{g}{R} & 0 \end{pmatrix}.$$

The eigenvalues are $\pm\sqrt{\omega^2 + g/R}$. Unlike in the previous case, the term under the square root sign is always positive, so the eigenvalues are always real and have opposite signs. This means that this point is an unstable saddle point.

This point is the top of the hoop. This is an unstable critical point no matter how fast the hoop rotates, because if the marble moves away from the top of the hoop, it is both pushed down by gravity and pushed further out by the centrifugal force.

The critical points $(\pm\cos^{-1}(g/R\omega^2), 0)$**:** First note that

$$\cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1.$$

This lets us evaluate the Jacobian at these points. It is

$$\begin{pmatrix} 0 & 1 \\ \omega^2(2(g/R\omega^2)^2 - 1) - (g/R)(g/R\omega^2)0 & \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g^2}{R^2\omega^2} - \omega^2 & 0 \end{pmatrix}.$$

The eigenvalues are $\pm\sqrt{g^2/R^2\omega^2 - \omega^2}$. These are real iff

$$g^2/R^2\omega^2 > \omega^2.$$

But this is equivalent to

$$g/R\omega > \omega$$

or

$$g/R\omega^2 > 1.$$

And we already said that these critical points only exist if $g/R\omega^2 < 1$. Thus, if the critical points exist, the corresponding eigenvalues are complex conjugates with zero real part. So these points are centers.

These are the most surprising critical points of the system. They are the points at which the outward centrifugal force is balanced by the downward gravitational force. Note that they are between $-\pi/2$ and $\pi/2$, i. e., always on the bottom half of the hoop. As our calculation shows, a marble that starts near these points will oscillate around them.

(b) *At each center of the system, find the period of oscillations of the linearization. This is approximately the period of small oscillations of the nonlinear system around that point.*

Case I: $g/R\omega^2 > 1$. In this case, $(0, 0)$ is the only center of the system. The eigenvalues are the complex conjugate pair $\pm\sqrt{\omega^2 - g/R}$, so the solutions to the linearization at this point look like

$$x = C_1 \cos(\sqrt{g/R - \omega^2}t) + C_2 \sin(\sqrt{g/R - \omega^2}t).$$

These oscillate with angular frequency $\sqrt{g/R - \omega^2}$ and period

$$\frac{2\pi}{\sqrt{g/R - \omega^2}}.$$

Observe that the period is smallest when $\omega = 0$, in which case it is $2\pi\sqrt{R}/\sqrt{g}$ – the same as for a pendulum of length R. As ω approaches g/R from below, the period goes to ∞.

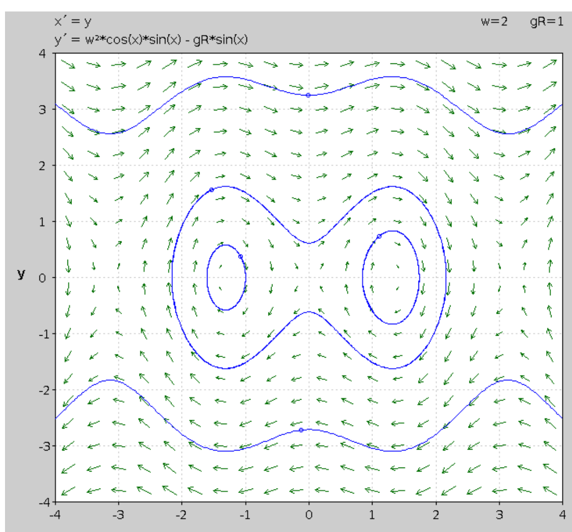
Case II: $g/R\omega^2 < 1$. Here the centers are the points $(\pm\cos^{-1}(g/R\omega^2), 0)$. By the same argument, the period is

$$\frac{2\pi}{\sqrt{\omega^2 - g^2/R^2\omega^2}} = \frac{2\pi}{\omega\sqrt{1 - (g/R\omega^2)^2}}.$$

As $\omega \to \infty$, the period goes to zero, and as $g/R\omega^2 \to 1$, the period goes to ∞.

(c) *If ω is large compared to g/R, there is another kind of closed orbit that doesn't just circle around a single center. You should be able to find this orbit in* ***pplane****. What is the marble doing in these orbits?*

I'm talking about the two-bulged orbit in this picture (made with $\omega = 2$ and $g/R = 1$):



You can see that it encloses the saddle point at $x = 0$ and the two centers, which are here at $x = \pm\pi/3$. So the marble must be oscillating around both centers. It starts to the right of the $x = \pi/3$ center, with enough initial acceleration due to gravity to fall past it and towards the bottom of the hoop. Note that it slows down as it passes the bottom of the hoop, as gravity is unable to accelerate it there since it needs to move sideways. However, after it passes the bottom at $x = 0$, the centrifugal force accelerates it in the same direction, with enough energy to pass $x = -\pi/3$. From there, the cycle repeats in the other direction.

(d) *Use Matlab and `ode45` to find the period of these more complicated orbits, for some different choices of the parameters and initial values. Either come up with an approximate formula for these periods, or just describe how they depend on the parameters.*

Let's call these orbits **double orbits**. It helps guide our investigation if we set some constraints on it first.

- There are really just two parameters: ω and g/R. Double orbits only happen if $\omega > g/R$.
- I'll constrain myself to choosing initial velocity equal to 0, and just vary the initial positions and the parameters.
- By symmetry, I only have to think about positive initial values of x.
- The initial values of x have to be less than π, and they have to be at least. . . something. If x starts out close to the center $x = \cos^{-1}(g/R\omega)$, then x will orbit that center. But it's not immediately clear where the transition to double orbits happens. (We could probably figure it out by thinking about energy, since to have a double orbit, the marble needs enough potential energy to reach the bottom of the hoop.

We should also think about how to recognize these orbits in Matlab. These orbits are differentiated from the orbits around the single centers by the fact that x passes zero.

I saved this as `marblehoop.m`:

```
function fout = marblehoop(t,x)
global omega;
global gR;
fout = zeros(2,1);
fout(1) = x(2);
fout(2) = (omega^2*cos(x(1)) - gR)*sin(x(1));
end
```

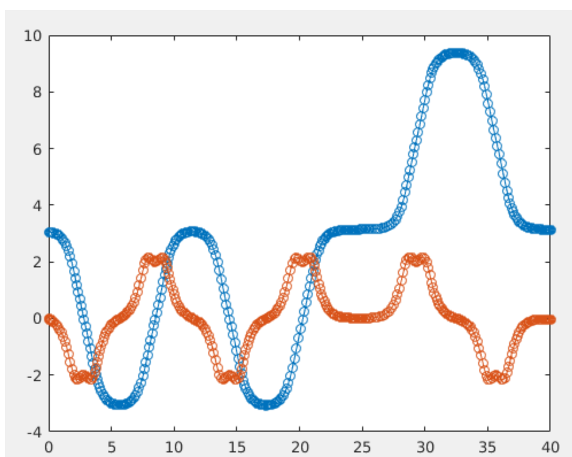
The "global" lines let me redefine the variables easily, by entering `global omega; global gR;` at the command prompt and then reassigning the variables from the command prompt whenever I want to.

Here are some observations I made. What did you see?

- For $g/R = 1$ and 2, and for various values of ω, I found the minimum $x(0)$ for a double orbit (call this x_{min}), the period of double orbits near $x(0) = x_{min}$, and the period of orbits near $x(0) = \pi$. Because I had technical problems getting close to π, "near π" meant exactly 3.
- Increasing either ω or g/R tended to decrease the periods and increase x_{min}.
- Double orbits near π had smaller periods than double orbits near x_{min}.
- For fixed g/R and varying ω, it seemed that the position of x_{min} was given by an arccosine function. I hypothesize that

$$x_{min} = 2\cos^{-1}\left(\frac{\sqrt{g/R}}{\omega}\right),$$

  though I've only checked this for two values of g/R.
- The periods near π appeared to be inversely proportional to ω for fixed g/R, though I was unable to get a good formula.
- Surprisingly (to me), Matlab was pretty bad at handling this system at even moderate levels of precision. Here is a graph I got for parameters `omega = 1.5; gR = 1;` and with the command `ode45('marblehoop', [0, 40], [3.05, 0])`.



In this graph, the marble does a double orbit twice and then does a third double orbit in the opposite direction, i. e., it has crossed the top of the hoop. It seems to me that this shouldn't happen just by a consideration of potential energy, so it must be the result of imprecision on Matlab's part. This happened any time I got too close to the minimum initial position for a double orbit, or to π.