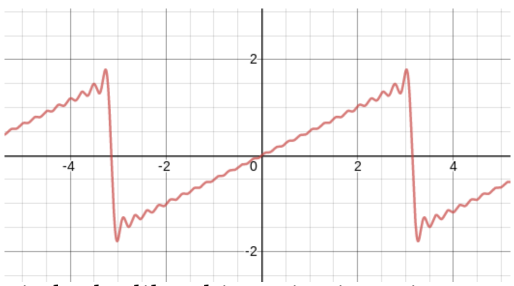
Math 303, Homework 6 solutions

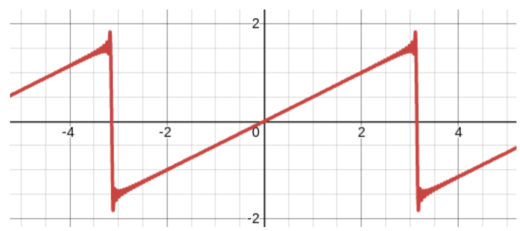
1. (a) *The Fourier series*

$$\sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) - \frac{1}{4}\sin(4t) + \cdots$$

converges to a 2π-periodic function with a much simpler description. Figure out what function it is, and prove your claim.

The sums of the first 25 terms, and the first 100 terms, in the Fourier series, look like this:





So it looks like this series is trying to converge to the 2π-periodic extension of a linear function. The line goes through the origin, so we just have to find its slope, which we can do by plugging in numbers. Since the convergence is better towards the middle of the line segment than near the points of discontinuity, it's best to pick x-values close to 0. By the way, Desmos has a nice way of doing this: if you write

$$f(x) = \sum_{n=1}^{100} \frac{(-1)^{(n+1)}}{n}\sin(nx)$$

and then type

$$f(1)$$

in another box, Desmos returns

$$= 0.50016\ldots.$$

So we could guess that $f(t)$ is the 2π-periodic function equal to $t/2$ on the interval $(-\pi, \pi)$.

Let's check this guess. Let $f(t)$ be this function, and let a_n and b_n be its Fourier coefficients. Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\,dt = \frac{1}{\pi}\left[\frac{t^2}{4}\right]_{-\pi}^{\pi} = 0.$$

For $n > 0$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\cos(nt)\,dt = \frac{1}{\pi}\left(\left[\frac{t}{2}\frac{\sin(nt)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{2}\frac{\sin(nt)}{n}\,dt\right)$$

(using integration by parts),

$$= 0 - \frac{1}{2\pi n}\left[-\frac{\cos(nt)}{n}\right]_{-\pi}^{\pi} = 0.$$

By the same argument,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\sin(nt)\,dt = \frac{1}{\pi}\left(\left[\frac{t}{2}\frac{-\cos(nt)}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{2}\frac{\cos(nt)}{n}\,dt\right).$$

Note that $\cos(n\pi) = (-1)^n$, so the first term simplifies to $\frac{(-1)^{n+1}}{n}$. The second term is

$$\frac{1}{2\pi n}\left[\frac{\sin(nt)}{n}\right]_{-\pi}^{\pi} = 0.$$

So we have $a_n = 0$ and $b_n = (-1)^{n+1}/n$, which are the same Fourier coefficients as in the series. So the given series is in fact the Fourier series of $f(t)$. Since the function $f(t)$ is piecewise smooth, the series converges to it away from the points of discontinuity, by the convergence theorem.

How else could we go about this problem? Graphing and guessing is by far the easiest way. It's also possible to evaluate the series at various points, say in MATLAB, and formulate a guess that way. If you do a lot of exercises like this, you start to build up a feel for what functions have what Fourier series, and you might at least be able to guess that the function is piecewise linear this way – though this is for experts only. Finally, if you know that the function is linear (or rather a periodic extension of a linear function), you could compute the Fourier series of the 2π-periodic extension of $f(t) = At+B$, write the coefficients in terms of A and B, and compare.

(b) *By evaluating your result at an appropriate number, come up with a cool formula of the form*

$$\pi/4 = \text{(something)}.$$

By the convergence theorem,

$$\frac{t}{2} = \sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) - \cdots \quad \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Plugging in $t = \pi/2$ gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

This is pretty surprising. The left-hand side is a geometrical quantity Archimedes could've thought about: the ratio of the circumference of a circle to the perimeter of a circumscribed square. To my knowledge, there's no "geometrical" proof that this is equal to the infinite series on the right (but maybe you can find one!). So one of the applications of Fourier series to math is as recipes for producing infinite series like this.

By the way, the series is not very efficient at calculating π. The sum of the first hundred terms is gives

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots - \frac{1}{199}\right) = 3.131592\cdots .$$

According to Desmos, it takes a thousand terms to get the first two decimal places ring.

2.

Come, investigate loneliness!
a solitary leaf
clings to the Kiri tree

–Basho

This leaf can be modelled as an undamped spring with mass 1 g and spring constant 0.5 g/s^2. Every two seconds, a dewdrop of mass 0.01 g lands on the leaf, and remains there for 1 second before sliding off. What (approximately) is the furthest that the leaf is displaced from equilibrium?

I believe the easiest way to do this problem is to treat the weight of the dewdrops as a time-dependent force acting on the leaf. Let x be the distance of the leaf from equilibrium, measured so that upward x is positive, and measured in meters, and let time be measured in seconds. If the leaf is left to move on its own, it satisfies an equation of motion

$$mx'' = -kx, \quad m = 1\,\mathrm{g}, \quad k = 0.5\,\mathrm{g/s^2}.$$

Recall that the right-hand side of this equation is actually the sum of the spring force and the weight of the leaf – we don't see a term mg in the equation because we've chosen $x = 0$ to be the point where this cancels the spring force. Writing $F(t)$ for the force on the leaf due to the weight of the dewdrop at time t (measured in millinewtons, i.e. g·m/s^2), we have

$$mx'' = -kx + F(t).$$

So what is $F(t)$? Say that there is a dewdrop on the leaf from $t = 0$ to $t = 1$, from $t = 2$ to $t = 3$, and so on. Then in these intervals of time, the force is 0.01 grams times g downwards, or -0.098 g·m/s^2. In the other intervals, the force is 0. So

$$F(t) = \text{ the 2-periodic extension of } \begin{cases} -0.098 & 0 \le t < 1 \\ 0 & 1 \le t < 2. \end{cases}$$

To solve the differential equation, we need to write F as a Fourier series. Note that the period is $2L = 2$, so the factor of $1/L$ that appears in the formulas for the Fourier coefficients is simply 1. Letting a_n and b_n be the Fourier coefficients, we have

$$a_0 = \int_0^2 F(t)\,dt = \int_0^1 -0.098\,dt = -0.098.$$

For $n > 0$,

$$a_n = \int_0^2 F(t)\cos(n\pi t)\,dt = \int_0^1 -0.098\cos(n\pi t)\,dt = -0.098\left[\frac{\sin(n\pi t)}{n\pi}\right]_0^1 = 0.$$

Finally,

$$b_n = \int_0^2 F(t)\sin(n\pi t)\,dt = \int_0^1 (-0.098)\sin(n\pi t)\,dt = 0.098\left[\frac{\cos(n\pi t)}{n\pi}\right]_0^1.$$

Note that $\cos(0) = 1$, while $\cos(n\pi) = (-1)^n$. Thus, the term in the brackets is $-2/(n\pi)$ if n is odd and 0 if n is even. So we have

$$b_n = \frac{-0.098(1-(-1)^n)}{n\pi} = \frac{-0.196}{n\pi} \text{ if } n \text{ is odd, } 0 \text{ if } n \text{ is even.}$$

In conclusion,

$$F(t) \sim -\frac{0.098}{2} + \sum_{n \text{ odd} \ge 1}^{\infty} \frac{-0.196}{n\pi}\sin(n\pi t).$$

And since F is piecewise smooth, this series actually converges to it away from the points of discontinuity. We can check our work here by graphing a partial sum on Desmos.

Now we solve the equation. Recall from class that the steady-state solution to

$$x'' + \omega_0^2 x = A\sin(\omega t), \quad \omega \ne \omega_0$$

is

$$x = \frac{A}{\omega_0^2 - \omega^2}\sin(\omega t).$$

(And if you don't recall this, you should try to recall how we found it, which is more important.) Also, the equation

$$x'' + \omega_0^2 x = A$$

has the constant solution

$$x = \frac{A}{\omega_0^2}.$$

We are dealing with the equation

$$x'' + 0.5x = -\frac{0.098}{2} + \sum_{n \text{ odd} \geq 1}^{\infty} \frac{-0.196}{n\pi} \sin(n\pi t).$$

Consider the equations obtained by replacing the right-hand side of this by a single one of its terms, i. e.

$$\begin{aligned}
x'' + 0.5x &= -\frac{0.098}{2} & &\rightsquigarrow x = -0.098, \\
x'' + 0.5x &= -\frac{0.196}{\pi} \sin(\pi t) & &\rightsquigarrow x = -\frac{0.196}{\pi(0.5^2 - \pi^2)} \sin(\pi t), \\
& & &\vdots \\
x'' + 0.5x &= -\frac{0.196}{n\pi} \sin(n\pi t) & &\rightsquigarrow x = -\frac{0.196}{n\pi(0.5^2 - n^2\pi^2)} \sin(n\pi t).
\end{aligned}$$

By adding together all these steady-state solutions, we obtain a Fourier series for a 2-periodic solution to the equation, namely

$$x = -0.098 + \sum_{n \text{ odd} \geq 1} -\frac{0.196}{n\pi(0.5^2 - n^2\pi^2)} \sin(n\pi t).$$

Here is my graph of the 100th partial sum of the Fourier series for the forcing function (in purple) and the steady-state response $x(t)$ (in red).

Now, the question I asked is underspecified – I didn't give you initial conditions for the motion! Using the general solution

$$x = -0.098 + \sum_{n \text{ odd} \geq 1} -\frac{0.196}{n\pi(0.5^2 - n^2\pi^2)} \sin(n\pi t) + C_1 \cos(\sqrt{0.5}t) + C_2 \sin(\sqrt{0.5}t),$$

you could choose your own initial conditions and answer the question there. One natural assumption is $C_1 = C_2 = 0$ – this means considering just the steady-state behavior of the leaf. In this case, we can see from Desmos that the displacement oscillates around $x = 0.098$ meters below equilibrium, with maximum displacement $\boxed{x = 0.104 \text{ meters}}$. It makes sense that the center of the oscillations is below the equilibrium, since the dewdrops exert a force which is down, on average. Note also that x is nearly a sine wave, since the $n = 1$ term in its Fourier series is much larger than any of the others.

Another natural assumption is $x(0) = x'(0) = 0$. The condition $x(0) = 0$ means that $C_1 = 0.098$. Differentiating the Fourier series termwise gives

$$x' = \sum_{n \text{ odd} \geq 1} -\frac{0.196}{0.5^2 - n^2\pi^2} \cos(n\pi t) - \sqrt{0.5}C_1 \sin(\sqrt{0.5}t) + \sqrt{0.5}C_2 \cos(\sqrt{0.5}t),$$

so

$$x'(0) = \sum_{n \text{ odd} \geq 1} -\frac{0.196}{0.5^2 - n^2\pi^2} + \sqrt{0.5}C_2.$$

So C_2 can only be calculated numerically.