Math 303, Homework 7

Due October 17, 2019

1. **Parseval's identity** states that, if f is a piecewise smooth and $2L$-periodic function with Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right),$$

then

$$\frac{1}{L}\int_{-L}^{L} f(t)^2 \, dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right).$$

Without worrying too much about convergence issues – that is, you can rearrange terms in infinite series, integrate them term by term, and so on, as much as you'd like – give a justification for this statement. *(Intuitively, this says that if f doesn't attain very large values, then its Fourier coefficients aren't very large, and vice versa.)*

2. Write down the Fourier series for the 2π-periodic function equal to t/π on the interval $(-\pi, \pi)$. Using Parseval's identity, prove that

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

3. The Fourier series

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos(3^n t)}{2^n}$$

converges to a continuous function. Using the theorem about differentiating Fourier series that we learned in class, show that f does not have a piecewise smooth first derivative.

Remarks for those who like pure math. The series

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \quad \text{for } s > 1,$$

is usually called $\zeta(s)$. In problem 2, you calculated $\zeta(2)$. You can calculate $\zeta(4)$, $\zeta(6)$, and so on by applying the same method to periodic extensions of other polynomial functions (try it out!). In fact, $\zeta(2k)$, for integer k, is always a rational multiple of π^{2k}. On the other hand, the series $\zeta(2k+1)$ are very poorly understood. While the calculation of $\zeta(2)$ was known to Euler in the 18th century, the fact that $\zeta(3)$ is *irrational* was only proved in the 1970s!

The series

$$\sum_{n=1}^{\infty} a^n \cos(b^n t),$$

where $0 < a < 1$ and b is sufficiently large, is known as the **Weierstrass monster**. If b is an integer, this is actually a Fourier series. Weierstrass proved a much stronger statement than problem 3: that if $ab \geq 1 + \frac{3\pi}{2}$, then the sum of the series is continuous at *every point* and differentiable at *no point*. This shocking result challenged the mathematical world's intuitions about how continuous functions worked and forced it to revisit the very foundations of analysis.