Math 303, Homework 7 solutions

1. ***Parseval's identity*** *states that, if f is a piecewise smooth and $2L$-periodic function with Fourier series*

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right),$$

then

$$\frac{1}{L}\int_{-L}^{L} f(t)^2\,dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Without worrying too much about convergence issues – that is, you can rearrange terms in infinite series, integrate them term by term, and so on, as much as you'd like – give a justification for this statement.

Since f is piecewise smooth, the Fourier series converges to it on the interval $[-L, L]$, except at its finitely many discontinuities. So the integral of $f(t)^2$ is the same as the integral of the square of the Fourier series. This square is

$$\frac{a_0^2}{4} + \sum_{n=1}^{\infty}\left(\frac{a_0 a_n}{2}\cos\left(\frac{n\pi t}{L}\right) + \frac{a_0 b_n}{2}\sin\left(\frac{n\pi t}{L}\right)\right)$$
$$+ \sum_{m,n=1}^{\infty}\left(a_n a_m \cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) + a_n b_m \cos\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right)\right.$$
$$\left. + b_n a_m \sin\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) + b_n b_m \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right)\right).$$

Let's integrate these terms one at a time.

The constant term:

$$\frac{1}{L}\int_{-L}^{L}\frac{a_0^2}{4}\,dt = \frac{1}{L}\frac{a_0^2}{4}(2L) = \frac{a_0^2}{2}.$$

The cosine terms:

$$\frac{1}{L}\int_{-L}^{L}\frac{a_0 a_n}{2}\cos\left(\frac{n\pi t}{L}\right)\,dt = \frac{1}{L}\left[\frac{a_0 a_n}{2}\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right]_{-L}^{L}.$$

This is zero because $\sin(-n\pi) = \sin(n\pi) = 0$.

The sine terms:

$$\frac{1}{L}\int_{-L}^{L} \frac{b_0 b_n}{2} \sin\left(\frac{n\pi t}{L}\right) dt = \frac{1}{L}\left[-\frac{b_0 b_n}{2}\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)\right]_{-L}^{L}.$$

This is zero because $\cos(-n\pi) = \cos(n\pi)$ (it's either 1 or -1, but they're the same).

The products of sines and cosines: There's one of these with an $a_n b_m$ in front and one with a $b_n a_m$. But in either case, all we need to check is

$$\int_{-L}^{L} \cos(n\pi t/L)\sin(m\pi t/L)\, dt = 0,$$

for any choice of n and m. The simplest way to do this is to notice that the cosine function is even and the sine function is odd, so their product is odd, so its integral along the symmetric interval $[-L, L]$ is zero. Another way is to recall that we showed this in class (it's also in the book) at the beginning of our study of Fourier series. We really just proved that

$$\int_{-\pi}^{\pi} \cos(nt)\sin(mt)\, dt = 0,$$

but by changing variables, say $u = \pi t/L$, one can reduce the first integral to the second. Finally, you can actually calculate either integral by starting with the trig identity

$$\cos\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right) = \frac{1}{2}\left(\sin\left(\frac{(m+n)\pi t}{L}\right) + \sin\left(\frac{(m-n)\pi t}{L}\right)\right).$$

The products of cosines and cosines, and sines and sines: Now we need to use the formula

$$\int_{-L}^{L} \cos(n\pi t/L)\cos(m\pi t/L)\, dt = \begin{cases} L & n = m \\ 0 & n \neq m. \end{cases}$$

Again, there are a number of ways to get this formula. You can quote it from the book; use the formula with $L = \pi$ we checked in class, and do a u-substitution with $u = \pi t/L$; or prove it directly from the appropriate trig identity. Here the necessary identity is

$$\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) = \frac{1}{2}\left(\cos\left(\frac{(m+n)\pi t}{L}\right) + \cos\left(\frac{(m-n)\pi t}{L}\right)\right).$$

If $m = n$, then the second term becomes just 1, which behaves differently than an ordinary cosine function when integrated. This explains the two cases.

Analogous remarks apply to the formula

$$\int_{-L}^{L} \sin(n\pi t/L)\sin(m\pi t/L)\,dt = \begin{cases} L & n = m \\ 0 & n \neq m. \end{cases}$$

To sum up, in the enormous series formula for $f(t)^2$ above, the only terms which have a nonzero integral are the constant term and the terms

$$a_n^2 \cos^2\left(\frac{n\pi t}{L}\right) \quad \text{and} \quad b_n^2 \sin^2\left(\frac{n\pi t}{L}\right).$$

Using the integrals we just did, we get that

$$\int_{-L}^{L} f(t)^2\,dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

If you're curious, the technical issues that I told you not to care about are: even if the Fourier series converges to $f(t)$, does its square converge to $f(t)^2$? And can we integrate the square of the series term by term? These problems can usually be ignored if the series converges fast enough – a good rule of thumb is that you can do whatever you want to a series whose terms decrease at least as fast as $1/n^2$. But if you want to know more about what can go wrong, sign up for a real analysis class!

2. *Write down the Fourier series for the 2π-periodic function equal to t/π on the interval $(-\pi, \pi)$. Using Parseval's identity, prove that*

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

Let's write $f(t)$ for the periodic function. This function is odd, so we know automatically that all the a_n's are zero. We then have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{\pi}\sin(nt)\,dt.$$

Integrating by parts, with $u = \frac{t}{\pi}$ and $v' = \sin(nt)$, gives

$$b_n = \frac{1}{\pi}\left[\frac{t}{\pi}\frac{(-\cos(nt))}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\pi}\frac{(-\cos(nt))}{n}\,dt = \frac{-2\cos(n\pi)}{n\pi}.$$

The right-hand integral is zero because the integral of a cosine function over an integer number of periods is always zero.

So we have

$$f(t) \sim \sum_{n=1}^{\infty} \frac{-2\cos(n\pi)}{n\pi} \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin(n\pi t).$$

The function f is piecewise smooth, so we can apply Parseval's identity. This says that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2 \, dt = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}.$$

But on the interval $(-\pi, \pi)$, f is just equal to t/π. So the left-hand side is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{t^2}{\pi^2} \, dt = \frac{1}{\pi} \left[\frac{t^3}{3\pi^2} \right]_{-\pi}^{\pi} = \frac{2}{3}.$$

So we have

$$\frac{2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}.$$

Multiplying both sides by $\pi^2/4$ gives

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is what we wanted to prove.

(For a completely different proof of this, look in section 9.2 of the book, which proves it as an application of the Fourier series convergence theorem!)

3. *The Fourier series*

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos(3^n t)}{2^n}$$

converges to a continuous function. Using the theorem about differentiating Fourier series that we learned in class, show that f does not have a piecewise smooth first derivative.

Suppose that f does have a piecewise smooth first derivative. We'll try to find a contradiction.

The **differentiation theorem** says that, if f is continuous (which I told you in the problem) and if f' is piecewise smooth (which we are assuming), then the Fourier series for f' is given by the termwise derivative of the Fourier series for f. So

$$f'(t) \sim \sum_{n=1}^{\infty} -\frac{3^n \sin(3^n t)}{2^n}.$$

The **convergence theorem** says that, if f' is piecewise smooth, then its Fourier series converges at every point. More precisely, it converges to $f'(t)$ if f' is continuous at t, and to $\frac{1}{2}(f'(t+) - f'(t-))$ if f' is discontinuous at t. (Note that $f'(t+)$ and $f'(t-)$ are both finite, by definition of "piecewise smooth").

But it's just not true that the Fourier series we found for f' converges at every point. In fact, it appears to converge at almost no point. You can see evidence for this if you try graphing a partial sum (even up to like $n = 5$), or calculating the value of f' on WolframAlpha or Matlab. This isn't a proof, because it could be true that the sum converges to a number too big for these programs to handle, or that, say, the $n = 1000$ term comes along and wipes everything out. But it is an acceptable argument in this class.

For a rigorous proof, let's calculate the Fourier series for f' at $t = \pi/2$. Since 3^n is always odd, $\sin(3^n\pi/2)$ is either 1 or -1. So the absolute value of the nth term in the series for $f'(\pi/2)$ is $3^n/2^n$. This clearly grows with n. But if the absolute values of the terms in a series don't go to 0 with increasing n, then the series diverges. So the series diverges at $t = \pi/2$, which is the contradiction we wanted.

(Since our applications of the two theorems depended on a hypothesis we now know is false – that f' is piecewise smooth – we shouldn't believe the conclusions we got from them, either. In particular, it's *just not true* to say that f' is given by the Fourier series

$$f'(t) \sim \sum_{n=1}^{\infty} -\frac{3^n \sin(3^n t)}{2^n}.$$

I would guess – this is Weierstrass's theorem if we replace the number 3 with a bigger one – that f' just doesn't exist for most, if not all, values of t, so it doesn't make sense to talk about its Fourier series at all. In this case, differentiating term by term just gives us divergent nonsense.)