# Math 303, Homework 8 solutions

1. *This problem only requires trig, but I've alluded to it often enough in class that you might as well do it. Suppose that you have a linear combination of cosines and sines of the same frequency, of the form*

$$A\cos(\omega t) + B\sin(\omega t).$$

*Show that this can be written as a single, phase-shifted sine function, of the form*

$$R\sin(\omega t + \delta).$$

We want to find an expression of the form

$$A\cos(\omega t) + B\sin(\omega t) = R\sin(\omega t + \delta).$$

It's easiest to start with the right-hand side. Using the formula for sine of a sum of angles,

$$R\sin(\omega t + \delta) = R\sin(\omega t)\cos(\delta) + R\cos(\omega t)\sin(\delta).$$

For this to equal $A\cos(\omega t) + B\sin(\omega t)$, we must have

$$R\cos(\delta) = A, \quad R\sin(\delta) = B.$$

Thus,

$$A^2 + B^2 = R^2(\cos^2\delta + \sin^2\delta) = R^2,$$

so assuming $R$ is positive, we must have

$$R = \sqrt{A^2 + B^2}.$$

The phase shift $\delta$ is then uniquely determined by the equations

$$\sin(\delta) = A/R, \quad \cos(\delta) = B/R.$$

Note that we can divide the first equation by the second to get

$$\tan(\delta) = A/B.$$

It is tempting to now write $\delta = \tan^{-1}(A/B)$, but this isn't quite right. The inverse tangent function (by convention) outputs an angle in the interval $(-\pi/2, \pi/2)$, and $\delta$ may not lie in this interval. For any given value of $A/B$, there are really two angles $\theta$ with $\tan(A/B)$, namely $\tan^{-1}(A/B)$ and $\tan^{-1}(A/B) + \pi$. Since $\delta \in (-\pi/2, \pi/2)$ iff it has positive cosine iff $B > 0$ (since we chose $R > 0$), we can write

$$\delta = \begin{cases} \tan^{-1}(A/B) & B > 0 \\ \tan^{-1}(A/B) + \pi & B < 0. \end{cases}$$

There's an edge case when $B = 0$. In this case,

$$\sin(\delta) = A/R = \pm 1, \quad \cos(\delta) = 0.$$

In this case $\delta = \pi/2$ if $A$ is positive and $\delta = -\pi/2$ if $A$ is negative.

2. *Suppose that a spring-mass system with mass 1 kg, damping constant 1 kg/s, and spring constant 1 kg/s$^2$ is forced by a sinusoidal force $F(t) = \sin(\omega t)$. Here $\omega$ is left as a parameter, so your answers will depend on $\omega$.*

   (a) *Find a formula for the steady-state motion of the mass.*

   The equation to be solved is

$$x'' + x' + x = \sin(\omega t).$$

   Since we are looking for a steady-state solution, we assume that $x$ is oscillating with the same frequency as the forcing function, so

$$x = A\sin(\omega t) + B\cos(\omega t).$$

   Then

$$\begin{aligned} x' &= A\omega\cos(\omega t) - B\omega\sin(\omega t), \\ x'' &= -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t). \end{aligned}$$

   Putting these in the left-hand side of the equation gives

$$((-A\omega^2 - B\omega + A)\sin(\omega t) + (-B\omega^2 + A\omega + B)\cos(\omega t) = \sin(\omega t).$$

   For these two functions of $t$ to be equal for all values of $t$, the coefficients of $\sin(\omega t)$ have to be equal, and likewise the coefficients of $\cos(\omega t)$ have to be equal. So we have a system of linear equations

$$\begin{aligned} -A\omega^2 - B\omega + A &= 1, \\ -B\omega^2 + A\omega + B &= 0. \end{aligned}$$

The second equation gives

$$A = \frac{\omega^2 - 1}{\omega} B.$$

Substituting for $A$ in the first equation, we have

$$1 = \frac{(1-\omega^2)(\omega^2-1)}{\omega} B - \omega B = \frac{-(1-\omega^2)^2 - \omega^2}{\omega}.$$

Thus,

$$B = \frac{-\omega}{(1-\omega^2)^2 + \omega^2}$$

and

$$A = \frac{1-\omega^2}{(1-\omega^2)^2 + \omega^2}.$$

So the steady-state solution is

$$x = \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2} \sin(\omega t) - \frac{\omega}{(1-\omega^2)^2+\omega^2} \cos(\omega t).$$

(b) *Using the previous problem, find the amplitude and phase shift of the motion.*

The amplitude is

$$R = \sqrt{A^2 + B^2} = \sqrt{\frac{(1-\omega^2)^2 + \omega^2}{\left((1-\omega^2)^2+\omega^2\right)^2}} = \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}.$$

The phase shift $\delta$ satisfies

$$\tan(\delta) = A/B = \frac{1-\omega^2}{-\omega} = \frac{\omega^2-1}{\omega}.$$

If we assume that $\omega$ is positive (not a drastic simplification of the problem, as $\sin(-\omega t) = -\sin(\omega t)$), then $B$, which is $-\omega$ divided by a sum of squares, will be negative. Thus, the previous problem implies that

$$\delta = \tan^{-1}\left(\frac{\omega^2-1}{\omega}\right) + \pi.$$

(c) *What value of $\omega$ (exactly) gives a steady-state motion of maximum amplitude?*

Let us think of

$$R = \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}$$

as a function of $\omega$. To *maximize* $R$, we must *minimize* its denominator, which means minimizing

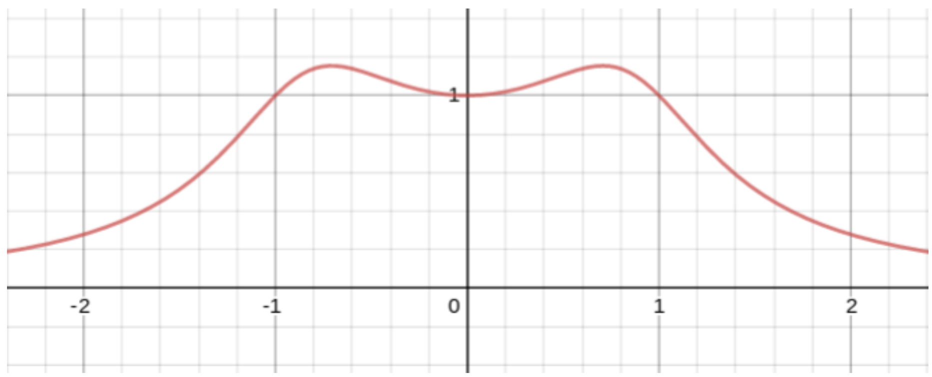
$$f(\omega) = (1-\omega^2)^2 + \omega^2.$$

We know from calculus that the minimum of a differentiable function can only occur at a point where its derivative is zero. We have

$$f'(\omega) = 2(-2\omega)(1-\omega^2) + 2\omega = -4\omega + 4\omega^3 + 2\omega = 4\omega^3 - 2\omega.$$

This vanishes when $\omega = 0$ or when $\omega = \pm\sqrt{1/2}$. By looking at a graph or calculating the second derivative, one sees that $\omega = 0$ is a local maximum of $f$ (so a minimum of $R$), so $\omega = \pm\sqrt{1/2}$ is the value we're interested in. If we keep the restriction that $\omega$ is positive, then just $\omega = \sqrt{1/2}$ is good enough.

Here's a graph of $R$ versus $\omega$:




*Remark*: If the spring were undamped but with the same $m$ and $k$ – so we were looking at equations of the form

$$x'' + x = \sin(\omega t)$$

– then pure resonance would occur when $\omega = 1$ (as you can check). This example shows that, if we think of the "resonant frequency" as the forcing frequency which maximizes the amplitude of the response, then the existence of damping can move the resonant frequency quite a bit ($\sqrt{1/2} \approx 0.71$). Here's another problem for you to think about: this example is underdamped, meaning $\gamma < 4km$ and the unforced motions are decaying oscillations; what happens in critically damped or overdamped cases? What forcing frequency maximizes the response amplitude then?

(d) *Suppose that the system in the previous problem is forced by the $\pi$-periodic square wave, defined on the interval $[0, \pi)$ by*

$$F(t) = \begin{cases} 1 & 0 \le t < \pi/2, \\ -1 & \pi/2 \le t < \pi. \end{cases}$$

*Find a formula for the steady-state motion of the mass.*

Let's first find a Fourier series for the square wave. Here the period is $\pi$ so $L = \pi/2$. Note that this wave is also an odd function (this is easiest to see if you graph it and extend it periodically). So its Fourier series has only sine terms, and we can write

$$b_n = \frac{2}{L} \int_0^L F(t) \sin(n\pi t/L)\, dt = \frac{4}{\pi} \int_0^{\pi/2} \sin(2nt)\, dt.$$

This integral is

$$\frac{4}{\pi} \left[ \frac{-1}{2n} \cos(2nt) \right]_0^{\pi/2} = \frac{2}{n\pi} \left(1 - \cos(n\pi)\right).$$

Thus,

$$F(t) \sim \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n\pi} \sin(2nt) = \frac{4}{\pi} \sin(2t) + \frac{4}{3\pi} \sin(6t) + \frac{4}{5\pi} \sin(10t) + \cdots.$$

(Since the square wave is piecewise smooth, $\sim$ means here that the series converges to $F$ at all values of $t$, except the ones where $F$ is discontinuous.)

So the equation we want to look at is

$$x'' + x' + x = \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n\pi} \sin(2nt).$$

Let $x_n$ be the steady-state solution to

$$x_n'' + x_n' + x_n = \sin(2nt).$$

Then, because the left-hand side of these differential equations is linear,

$$x = \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n\pi} x_n$$

will be a solution to our problem.

Finally, the equation defining $x_n$ is the same as the equation we solved in question 2, where $\omega = 2n$. So the steady-state solution is

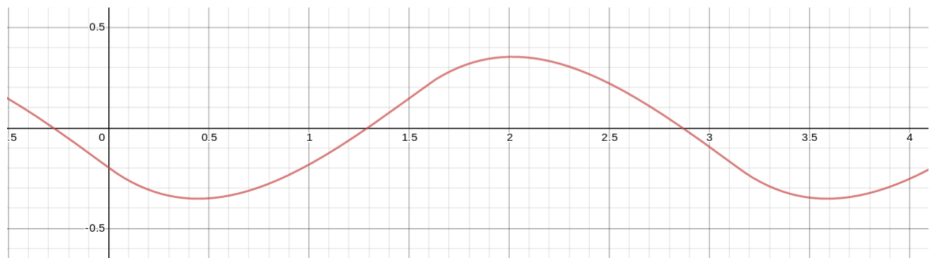
$$x_n = \frac{1-4n^2}{(1-4n^2)^2 + 4n^2} \sin(2nt) - \frac{2n}{(1-4n^2)^2 + 4n^2} \cos(2nt).$$

Thus, the solution to the problem is

$$x = \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n\pi} \left( \frac{1-4n^2}{(1-4n^2)^2 + 4n^2} \sin(2nt) - \frac{2n}{(1-4n^2)^2 + 4n^2} \cos(2nt) \right).$$

We should make two observations. First, since $b_2$, $b_4$, $b_6$ and so on are all zero, there are no contributions to $x$ of the form $\sin(4t)$, $\sin(8t)$, and so on. Second, the trig function inside the parentheses has the largest possible amplitude when $2n = \sqrt{1/2}$, by our work in problem 2. This would make $n \approx 0.35$, which is closest to the integer 1 (but not very close at all). Also, $b_n$ is largest when $n$ is 1. So we expect the $n = 1$ term, the one that looks like a linear combination of $\sin(2t)$ and $\cos(2t)$, to contribute the most to the actual behavior of $x$.

These observations help us understand the graph of $x(t)$:




which looks like a slightly lopsided, $\pi$-periodic sine wave of amplitude 0.3529. Note that the amplitude of just the $n = 1$ term of the series is

$$\frac{4}{\pi} \cdot \frac{1}{\sqrt{(1-4)^2 + 4}} \approx 0.3531,$$

which is very close to the amplitude of the entire function.