Math 303, Homework 9

Due November 7, 2019

You may be able to find more information about both these problems in the book. Both problems have to do with a heat-conducting rod, insulated along its length, of length L and thermal diffusivity K. Recall that the temperature distribution $u(x,t)$ of such a rod satisfies the heat equation:

$$u_t = Ku_{xx}. \tag{1}$$

1. Suppose that one end of the rod is held at temperature 0 and the other end is held at a nonzero fixed temperature H. So the boundary conditions are now

$$u(0,t) = 0, \quad u(L,t) = H. \tag{2}$$

   (a) Find a particular solution $u_p(x,t)$ to the heat equation that satisfies the boundary conditions (2). *(Hint: Don't try to use separation of variables – you just want one solution! Instead, see if you can make both sides of* (1) *equal to zero?)*

   (b) Let u be an arbitrary solution to the heat equation that satisfies (2), with initial temperature distribution

$$u(x,0) = f(x).$$

   Show that

$$v(x,t) = u(x,t) - u_p(x,t)$$

   satisfies the heat equation together with the simpler boundary conditions,

$$v(0,t) = 0, \quad v(L,t) = 0,$$

   and the initial condition

$$v(x,0) = f(x) - u_p(x,0).$$

   (c) As an example, find $u(x,t)$, subject to (2), if the initial temperature is constant:

$$u(x,0) = H \quad (0 < x \leq L).$$

2. Suppose that one end of the rod is held at temperature 0 and the other end is insulated:

$$u(0,t) = 0, \quad u_x(L,t) = 0. \tag{3}$$

(a) Show that, if $u(x,t) = X(x) \cdot T(t)$, then u is a scalar multiple of a function of the form

$$u_n(x,t) = \sin\left(\frac{n\pi}{2L}x\right) \exp(\left(\frac{-Kn^2\pi^2}{4L^2}t\right),$$

where n is *odd*.

(b) It follows that any function

$$u(x,t) = \sum_{n\geq 1 \text{ odd}} b_n \sin\left(\frac{n\pi}{2L}x\right) \exp(\left(\frac{-Kn^2\pi^2}{4L^2}t\right)$$

is a solution to the heat equation satisfying (3). The initial temperature distribution is

$$u(x,0) = \sum_{n\geq 1 \text{ odd}} b_n \sin\left(\frac{n\pi}{2L}x\right). \tag{4}$$

But this is only useful if we can write any (reasonable, i. e., piecewise smooth) function on $[0,L]$ in the form (4)!

Convince yourself that this is true, as follows. Let $f(x)$ be a function on $[0,L]$. Let $F(x)$ be the odd $4L$-periodic function such that

$$F(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(2L-x) & L \leq x \leq 2L. \end{cases}$$

Show that the Fourier series of $F(x)$ is of the form (4), and converges to $f(x)$ at all points on $[0,L]$ where $f(x)$ is continuous.

(c) Find $u(x,t)$, subject to (3), if the initial temperature distribution is constant:

$$u(x,0) = H \quad (0 < x \leq L).$$