Math 303, Homework 9

Due November 7, 2019

Recall that the temperature distribution $u(x,t)$ of such a rod satisfies the heat equation:

$$u_t = K u_{xx}. \tag{1}$$

1. *Suppose that one end of the rod is held at temperature* 0 *and the other end is held at a nonzero fixed temperature H. So the boundary conditions are now*

$$u(0,t) = 0, \quad u(L,t) = H. \tag{2}$$

   (a) *Find a particular solution $u_p(x,t)$ to the heat equation that satisfies the boundary conditions* (2). *(Hint: Don't try to use separation of variables – you just want one solution! Instead, see if you can make both sides of* (1) *equal to zero?)*

   I'll follow the hint. If $u_t = 0$, then $u(x,t)$ is a constant function of t for each value of x. In other words, it doesn't depend on t. It's only a function of x. Next, if $u_{xx} = 0$, then integrating once gives $u_x =$ a constant C_1, and integrating twice gives $u = C_1 x + C_2$. The only values of the constants that satisfy the boundary conditions are $C_1 = H/L$ and $C_2 = 0$. Thus,

$$u(x,t) = \frac{H}{L}x.$$

   As I mentioned in class, this is exactly the *steady state* of the bar, with the given boundary conditions. Over time, any temperature distribution will approach this one.

   (b) *Let u be an arbitrary solution to the heat equation that satisfies* (2)*, with initial temperature distribution*

$$u(x,0) = f(x).$$

   Show that

$$v(x,t) = u(x,t) - u_p(x,t)$$

   satisfies the heat equation together with the simpler boundary conditions,

$$v(0,t) = 0, \quad v(L,t) = 0,$$

and the initial condition

$$v(x, 0) = f(x) - u_p(x, 0).$$

We check the boundary conditions:

$$v(0, t) = u(0, t) - u_p(0, t) = 0 - 0 = 0,$$

$$v(L, t) = u(L, t) - u_p(L, t) = H - H = 0.$$

We check the heat equation:

$$v_t = u_t - (u_p)_t = Ku_{xx} - K(u_p)_{xx}$$

(since both u and u_p also satisfy the heat equation). But the right-hand side of this is just Kv_{xx}. Finally, the initial value of v is

$$v(x, 0) = u(x, 0) - u_p(x, 0) = f(x) - u_p(x, 0),$$

by definition of $f(x)$.

As you can see, this is really straightforward once you know what you're doing! It's an application of the linearity of the heat equation and the various boundary conditions. Using this linearity, we are able to replace the nonhomogeneous boundary condition $u(L, t) = H$ with a homogeneous one $u(L, t) = 0$.

(c) *As an example, find $u(x, t)$, subject to (2), if the initial temperature is constant:*

$$u(x, 0) = H \quad (0 < x \leq L).$$

Let $v(x, t) = u(x, t) - u_p(x, t)$. Then by part (b), v is a solution to the following problem:

$$\begin{aligned} v_t &= Kv_x x, \\ v(0, t) &= v(L, t) = 0, \\ v(x, 0) = u(x, 0) - u_p(x, 0) &= H - \frac{H}{L} x. \end{aligned}$$

Even though v isn't really a temperature distribution so much as a difference between two temperature distributions, this problem is mathematically identical to the one about a temperature distribution on a laterally insulated rod with ends held at temperature zero! By work we did in class, we know that $v(x, t)$ has the general form

$$v(x, t) = \sum b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-Kn^2\pi^2 t}{L^2}\right),$$

with initial condition

$$v(x, 0) = \sum b_n \sin\left(\frac{n\pi x}{L}\right).$$

Thus, b_n are the Fourier coefficients for the odd $2L$-periodic extension of the function $H - \frac{H}{L}x$. We have:

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L \left(H - \frac{H}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&\text{(integrate by parts with } u = H - Hx/L,\ v = \sin(n\pi x/L)) \\
&= \frac{2}{L}\left[\left(H - \frac{H}{L}x\right)\left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L - \frac{2}{L}\int_0^L \frac{H}{L}\cdot\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2H}{n\pi} - \frac{2H}{n\pi L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L \\
&= \frac{2H}{n\pi}.
\end{aligned}$$

This can be checked by graphing. Thus, for v, we get

$$v(x,t) = \sum_{n=1}^{\infty}\frac{2H}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-Kn^2\pi^2 t}{L^2}\right).$$

Since $u(x,t) = v(x,t) + u_p(x,t)$, the formula for u is

$$u(x,t) = \frac{H}{L}x + \sum_{n=1}^{\infty}\frac{2H}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-Kn^2\pi^2 t}{L^2}\right).$$

It's worth graphing this on Desmos to see how it behaves over time. The $x = 0$ end cools down very rapidly, and the area of the bar closer to $x = L$ cools much more slowly.

2. *Suppose that one end of the rod is held at temperature* 0 *and the other end is insulated:*

$$u(0,t) = 0, \quad u_x(L,t) = 0. \tag{3}$$

(a) *Show that, if $u(x,t) = X(x)\cdot T(t)$, then u is a scalar multiple of a function of the form*

$$u_n(x,t) = \sin\left(\frac{n\pi}{2L}x\right)\exp(\left(\frac{-Kn^2\pi^2}{4L^2}t\right),$$

where n is odd.

If $u = X\cdot T$, then the heat equation becomes

$$XT' = KX''T$$

or

$$\frac{X''}{X} = \frac{T'}{KT}.$$

Since the left-hand side is independent of t and the right-hand side is independent of x, both sides must be equal to the same constant, say $-\lambda$. Then the ordinary differential equation for X is

$$\frac{X''}{X} = -\lambda$$

or

$$X'' + \lambda X = 0. \tag{4}$$

We should also interpret the boundary conditions in terms of X and T:

$$X(0)T(t) = X'(L)T(t) = 0.$$

Since $T(t) = 0$ implies $u(x,t) = 0$, which is the trivial solution, we must have

$$X(0) = X'(L) = 0. \tag{5}$$

Now, there are three cases for (4), according to the sign of λ.

$\underline{\lambda < 0}$: Then the solutions for X are linear combinations of exponential functions,

$$X(x) = C_1 \exp(-\sqrt{-\lambda}x) + C_2 \exp(\sqrt{-\lambda}x).$$

Evaluating at $x = 0$ and using the boundary condition there gives

$$0 = X(0) = C_1 + C_2.$$

Likewise,

$$0 = X'(L) = -\sqrt{-\lambda}C_1 + \sqrt{-\lambda}C_2.$$

Since the coefficient vector $\langle -\sqrt{-\lambda}, \sqrt{-\lambda} \rangle$ isn't a scalar multiple of $\langle 1, 1 \rangle$, this system of equations for C_1 and C_2 has a single solution, which is clearly $C_1 = C_2 = 0$. So the only solution with $\lambda < 0$ is the trivial solution.

$\underline{\lambda = 0}$: Now we have

$$X(x) = C_1 + C_2 x,$$

and (5) implies

$$\begin{aligned} 0 &= X(0) = C_1, \\ 0 &= X'(L) = C_2. \end{aligned}$$

So the only solution in this case is also trivial.

$\underline{\lambda > 0}$: Here,

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

The boundary conditions imply

$$\begin{aligned} 0 &= X(0) = C_1, \\ 0 &= X'(L) = C_1\sqrt{\lambda}(-\sin(\sqrt{\lambda}L)) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L). \end{aligned}$$

Since $\lambda \neq 0$, $\cos(\sqrt{\lambda}L)$ must be zero. The zeros of the cosine function are of the form $n\pi/2$, where n is odd, and we only care about the positive solutions. Thus,

$$\sqrt{\lambda}L = \frac{n\pi}{2},\ n \text{ odd} > 0,$$

so

$$\lambda = \frac{n^2\pi^2}{4L^2},\ n \text{ odd} > 0.$$

Call this λ_n, so we have eigenvalues λ_1, λ_3, λ_5, ... The associated eigenfunctions are

$$X_n = \sin\left(\frac{n\pi x}{2L}\right),\ n \text{ odd} > 0.$$

(There's another reasonable way to write this – we could write

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \quad X_n = \sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad n = 1, 2, 3, \dots.$$

This would give us eigenvalues numbered λ_1, λ_2, and so on. Either way is fine, as long as we're consistent in what we call them, and we don't miss any eigenvalues or eigenfunctions.)

Returning to the original problem, we now have to solve the ODE for T:

$$\frac{T'}{KT} = -\lambda$$

or

$$T' = -\lambda KT.$$

Solutions to this are of the form

$$T(t) = C\exp(-\lambda Kt).$$

Taking $\lambda = \lambda_n$, and choosing the constant of integration to be 1, we get solutions

$$T_n = \exp(-\lambda_n Kt) = \exp\left(\frac{-Kn^2\pi^2 t}{4L^2}\right),\ n \text{ odd}.$$

These combine with the eigenfunctions X_n to give solutions to the problem of the form

$$u_n = X_nT_n = \sin\left(\frac{n\pi}{2L}x\right)\exp(\left(\frac{-Kn^2\pi^2}{4L^2}t\right),$$

for n odd, as was claimed.

(b) *It follows that any function*

$$u(x,t) = \sum_{n\geq 1 \text{ odd}} b_n \sin\left(\frac{n\pi}{2L}x\right)\exp(\left(\frac{-Kn^2\pi^2}{4L^2}t\right) \tag{6}$$

is a solution to the heat equation satisfying (3)*. The initial temperature distribution is*

$$u(x,0) = \sum_{n\geq 1 \text{ odd}} b_n \sin\left(\frac{n\pi}{2L}x\right). \tag{7}$$

But this is only useful if we can write any (reasonable, i. e., piecewise smooth) function on $[0,L]$ *in the form* (7)*!*

Convince yourself that this is true, as follows. Let $f(x)$ *be a function on* $[0,L]$*. Let* $F(x)$ *be the odd* $4L$*-periodic function such that*

$$F(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(2L-x) & L \leq x \leq 2L. \end{cases} \tag{8}$$

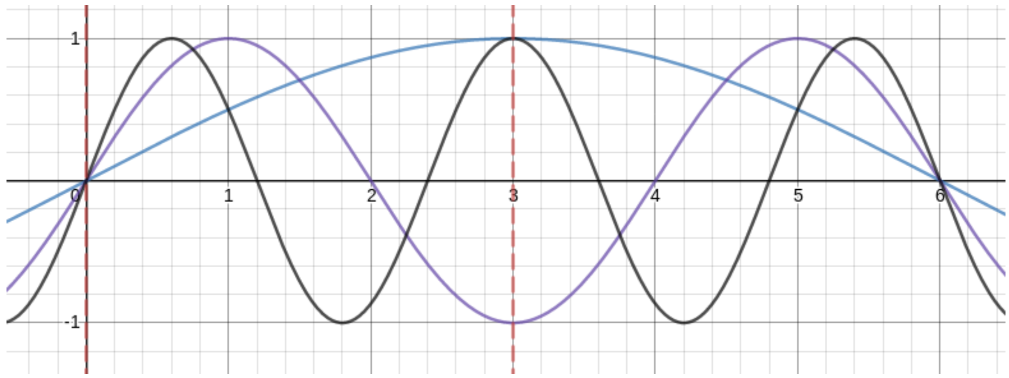
Show that the Fourier series of $F(x)$ *is of the form* (7)*, and converges to* $f(x)$ *at all points on* $[0,L]$ *where* $f(x)$ *is continuous.*

Initial remarks. Why is any function of the form (6) a solution to the heat equation and boundary conditions? Because it's a linear combination of the u_n, each of which solves the heat equation and boundary conditions; and because the heat equation and (3) are linear and homogeneous, meaning that they have the property that their sets of solutions are closed under linear combinations. (There's still something that could go wrong with an arbitrary function of the form (6), which is that it could fail to converge! But as long as the function $u(x,t)$ described there makes sense and is reasonable, we do get a solution to the heat equation and boundary conditions.)

Why is the initial temperature distribution of (6) the function of x described in (7)? We get this by substituting $t = 0$ into (6).

Can we just use the Fourier series for f to get the b_n's in (7)? The question doesn't quite make sense, as only periodic functions have Fourier series, and f, rather than being periodic, is only defined on the interval $[0,L]$. But we could take certain periodic extensions of f and take their Fourier series. For instance, we could take the L-periodic extension of f and get a Fourier series made of terms that look like $\cos(2n\pi x/L)$ and $\sin(2n\pi x/L)$. This probably isn't right, as the terms in (7) only have sines. Or we could take the odd $2L$-periodic extension of f and get a Fourier sine series, made up of terms of the form $\sin(n\pi x/L)$. But the frequencies $n\pi/L$ that show up here are different than the frequencies $n\pi/2L$, n odd, that show up in (7). So the series in (7) is a different kind of series. The point of the problem is that it's still a Fourier series, of a certain unusual periodic extension of f, and so it still has nice convergence properties, we can calculate the coefficients, etc.

Where does the idea for this strange $4L$-periodic extension come from? Here are the first few sine functions in the series (7), with $L = 3$ indicated by the dotted line:



All these functions have some properties in common. They're all odd (since they're all sine functions), they're all $4L$-periodic, and they're also all symmetric about the line $x = L$. So any linear combination of them must have the same properties. In particular, if we've figured out coefficients that express $f(x)$ as a sum of these functions on the interval $[0, L]$, then on the rest of the real line, that sum must be odd, $4L$-periodic, and symmetric about $x = L$. Now, if f has been specified on $[0, L]$, there's a unique extension of it to the whole real line that's odd, $4L$-periodic, and symmetric about $x = L$ (if you don't believe me, draw a graph of a random f and try to extend it elsewhere using these three properties!). This extension is the one defined by (8).

Solution to the problem: Let $F(x)$ be as described in (8). Let's try to describe the Fourier series of $F(x)$. Since F is odd and periodic, it has a Fourier sine series; since the period is $4L$, this series looks like

$$F(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2L}\right).$$

To show that this series is of the form (7), we need to show that $b_n = 0$ when n is even. Note that the only symmetry property of $F(x)$ that we *haven't* used so far is the fact that it's symmetric about $x = L$.

So suppose $n = 2k$ is even. Then b_{2k} is given by the integral

$$b_{2k} = \frac{2}{2L}\int_0^{2L} F(x)\sin\left(\frac{2k\pi x}{2L}\right)\,dx.$$

Now, the function $s(x) = \sin\left(\frac{2k\pi x}{2L}\right)$ is "odd about the line $x = L$", meaning that if we rotate its graph by 180 degrees around the point $(L, 0)$, we get the same graph. Equivalently, $s(2L-x) = -s(x)$. You can check this by looking at graphs, or by calculation

$$s(2L-x) = \sin\left(\frac{2k\pi(2L-x)}{2L}\right) = \sin\left(2k\pi - \frac{2k\pi x}{2L}\right) = \sin\left(-\frac{2k\pi x}{2L}\right) = -s(x)$$

because the sine function is odd. If we integrate any function with this property on the interval $[0, 2L]$, we get zero. This is because any positive contribution to the integral on one side of $x = L$ is cancelled out by an equal negative contribution on the opposite side. (Make a sketch if you don't believe me!) Now, since $F(x)$ is symmetric about $x = L$, $F(x)s(x)$ is also odd about $x = L$. Indeed,

$$F(2L - x)s(2L - x) = F(x)(-s(x)) = -F(x)s(x).$$

Thus, the integral giving b_{2k} vanishes, so $b_{2k} = 0$. This is what we wanted to prove.

(This hopefully seems familiar – it's a variant of the argument we used to show that the sine terms in the Fourier series for an even function, and the cosine terms in the Fourier series for an odd function, vanish.)

(c) *Find $u(x,t)$, subject to* (3)*, if the initial temperature distribution is constant:*

$$u(x, 0) = H \quad (0 < x \leq L).$$

Let's write

$$f(x) = u(x, 0) = H \quad (0 < x \leq L).$$

We need to write $f(x)$ in the form (7). But the previous step in the problem told us exactly how to do this – it's just the Fourier series of the associated $4L$-periodic function $F(x)$. Using the construction of (8), we have

$$F(x) = H \quad (0 < x \leq 2L), \quad F(x) \text{ odd and } 4L\text{-periodic.}$$

Since $F(x)$ is odd, its Fourier series is a Fourier sine series. We have

$$\begin{aligned} b_n &= \frac{2}{2L}\int_0^{2L} H \sin\left(\frac{n\pi x}{2L}\right) dx \\ &= \frac{1}{L}\left[-H\frac{2L}{n\pi}\cos\left(\frac{n\pi x}{2L}\right)\right]_0^{2L} \\ &= \frac{-2H}{n\pi}(\cos(n\pi) - 1) \\ &= \frac{2H}{n\pi}(1 - \cos(n\pi)). \end{aligned}$$

Note that this is 0 if n is even, as predicted. If n is odd, it's $4H/n\pi$.

So we have the series expansion

$$H = f(x) \sim \sum_{n\geq 1 \text{ odd}} \frac{4H}{n\pi}\sin\left(\frac{n\pi x}{2L}\right), \quad 0 < x \leq L.$$

The associated solution to the problem is

$$u(x,t) = \sum_{n\geq 1 \text{ odd}} \frac{4H}{n\pi}\sin\left(\frac{n\pi}{2L}x\right)\exp\left(\frac{-Kn^2\pi^2}{4L^2}t\right).$$