Math 303 Midterm 1 Review Sheet

Due September 19, 2019

1. **The meaning of a differential equation.**
   (a) You should know what it means for a function to solve a differential equation, or an initial value problem.
   (b) You should know how to translate a real-world phenomenon or word problem into a differential equation, and understand what the solutions to this equation tell you about the problem. There's no set curriculum for this, but you should especially be able to handle mechanical problems about springs and the like, and problems involving population dynamics. See sections 3.4, 5.4, and 6.3 of the book for some examples, as well as (ADD URL).
   (c) Some commonly occurring phenomena we've seen in class:
      (i) Functions of the form
      $$A\cos(\omega t), \quad A\sin(\omega t)$$
      oscillate with amplitude A and angular frequency ω, and with period $2\pi/\omega$. Any linear combination of the form
      $$A\cos(\omega t) + B\sin(\omega t)$$
      can be simplified to a function of the form
      $$R\cos(\omega t - \delta).$$
      (ii) Functions of the form
      $$e^{kt}$$
      grow exponentially if k is positive, or decay exponentially if k is negative. The function takes time $1/|k|$ to grow/shrink by a factor of e. The doubling time/half-life is $\ln(2)/|k|$.
      (iii) Functions of the form
      $$A\cos(\omega t)e^{kt}$$
      and so on oscillate and grow/decay simultaneously.

(iv) Functions of the form

$$P(t)e^{kt},$$

where P is a polynomial, behave similarly to e^{kt} after a sufficient amount of time passes.

(d) A system of nth-order differential equations can typically be turned into a first-order system, by introducing new variables for the first, ..., $(n-1)$th derivatives of the functions.

2. **Solving first-order homogeneous linear systems with constant coefficients.** (Sections 5.1, 5.2, and 5.5.)

(a) You should be able to solve systems of the form

$$\mathbf{x}' = A\mathbf{x}, \tag{1}$$

where $mathbf x$ is an $n \times 1$ vector-valued function and A is an $n \times n$ matrix of constants. I will only ask you to solve at most 2-dimensional systems by hand, but I may ask you questions about the process for higher-dimensional systems.

(b) The set of solutions to such an equation forms an n-dimensional vector space.

(i) That is, any linear combination of solutions is also a solution, and any linearly independent set of solutions has at most n elements.

(ii) If we turn (1) into an initial value problem by fixing the value of $\mathbf{x}(0)$, then this problem has a unique solution.

(iii) Given n functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$, we can check whether they're linearly independent by evaluating the Wronskian, which is the determinant of the matrix that has $\mathbf{x}_i$ for columns. The functions are linearly independent at some time t if their Wronskian is nonzero at t.

(iv) If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are solutions to (1), then their Wronskian is nonzero at some time t if and only if it is nonzero at all times t.

(c) To solve (1), first find the eigenvalues of A, which are the roots of the characteristic polynomial

$$\det(A - \lambda I).$$

There are n eigenvalues, counting repetitions.

(d) Then, for each eigenvalue λ, find an associated eigenvector. This is a solution $\mathbf{v}$ to the equation

$$A\mathbf{v} = \lambda\mathbf{v} \text{ or } (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

The set of eigenvectors of λ forms a vector space. If λ is repeated k times, then its space of eigenvectors has dimension between 1 and k inclusive.

(e) If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then the function

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

is a solution of (1).

(f) **Complex eigenvalues**:

(i) Assuming that A has real entries, any non-real eigenvalues will come in conjugate pairs, $\{\lambda, \overline{\lambda}\}$. If $\mathbf{v}$ is an eigenvector for λ, then $\overline{\mathbf{v}}$ is an eigenvalue for $\overline{\lambda}$.

(ii) The real and imaginary parts of the solution $\mathbf{v}e^{\lambda t}$ are both real-valued solutions. Thus, the pair of complex-valued solutions associated to λ and $\overline{\lambda}$ contribute a pair of real-valued solutions.

(iii) You can calculate the real and imaginary parts by expanding $e^{\lambda t}$ using Euler's formula:

$$e^{(a+bi)t} = e^{at}e^{bit} = e^{at}(\cos(bt) + i\sin(bt)).$$

(g) **Repeated eigenvalues**:

(i) If an eigenvalue repeated k times has k linearly independent eigenvectors, then there's nothing to worry about: you can use the above methods to find k linearly independent solutions associated to the eigenvalue.

(ii) Otherwise, you need to look for generalized eigenvectors, which are solutions to $(A-\lambda I)^r\mathbf{v} = 0$. The set of generalized eigenvectors forms a k-dimensional vector space.

(iii) Suppose $\mathbf{v}_1$ satisfies $(A-\lambda I)^r\mathbf{v}_1 = 0$ but not $(A-\lambda I)^{r-1}\mathbf{v}_1 = 0$. Let

$$\mathbf{v}_2 = (A-\lambda I)\mathbf{v}_1, \ldots, \mathbf{v}_r = (A-\lambda I)^{r-1}\mathbf{v}_1.$$

Then the following is a solution to (1):

$$\mathbf{x}(t) = \left(\frac{t^{r-1}}{(r-1)!}\mathbf{v}_r + \cdots + t\mathbf{v}_2 + \mathbf{v}_1\right)e^{\lambda t}.$$

(iv) In the 2×2 case, let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ. Let $\mathbf{w}$ satisfy

$$(A-\lambda I)\mathbf{w} = \mathbf{v}.$$

Then two linearly independent solutions are

$$\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t}, \quad \mathbf{x}_2(t) = (t\mathbf{v} + \mathbf{w})\, e^{\lambda t}.$$

3. **Nonlinear systems and critical point analysis.** (Sections 6.1 and 6.2.)

(a) The behavior of solutions to linear systems around the origin depends on the eigenvalues. See the table on the next page.

Name	How do the solutions behave?	When does it happen?
Source	Leave the origin as $t \to \infty$	Eigenvalues have real parts > 0
Sink	Approach the origin as $t \to \infty$	EIgenvalues have real parts < 0
Saddle point	Approach the origin along one line, then leave along another	One positive and one negative eigenvalue
Proper node	Approach/leave along all lines through the origin	Repeated nonzero eigenvalue with a full (2-dimensional) space of eigenvectors
Improper node	Approach/leave along one line	Real nonzero eigenvalues and not a proper node
Spiral point	Approach/leave along spirals	Complex eigenvalues with nonzero real parts
Center	Orbit the origin along ellipses	Complex eigenvalues with zero real part
Joker's trick	If zero is an eigenvalue, be careful and keep your wits about you	Zero is an eigenvalue
Stable	Solutions that start close to the origin stay close to the origin	Nodal or spiral sink, or center
Unstable	Not stable	Nodal or spiral source, or saddle point
Asymptotically stable	Solutions that start close to the origin approach the origin as $t \to \infty$	Nodal or spiral sink

(b) Now consider nonlinear systems of the form

$$\begin{aligned} x' &= F(x, y), \\ y' &= G(x, y). \end{aligned}$$

A **critical point** of the system is a point (x, y) where $x' = y' = 0$. The same sort of analysis can be used to describe behavior close to the critical point.

(c) The **Jacobian** of the system is the matrix

$$J(x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}.$$

(d) Let (x_c, y_c) be a critical point of the system. We say that the system is **almost linear** there if (i) F and G have continuous first partial derivatives at (x_c, y_c), (ii) (x_c, y_c) is an isolated critical point, and (iii) zero is not an eigenvalue of $J(x_c, y_c)$. In this case, the system has a Taylor expansion

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = J(x_c, y_c) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} r(u, v) \\ s(u, v) \end{pmatrix},$$

where $u = x - x_c$, $v = y - y_c$, and r and s are "remainder" functions satisfying

$$\lim_{(u,v)\to(0,0)} \frac{r(u, v)}{\sqrt{u^2 + v^2}} = \lim_{(u,v)\to(0,0)} \frac{s(u, v)}{\sqrt{u^2 + v^2}} = 0.$$

The **linearization** of the original system at (x_c, y_c) is the linear system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = J(x_c, y_c) \begin{pmatrix} u \\ v \end{pmatrix}.$$

(e) If the nonlinear system is almost linear, its linearization approximates it well near the critical point. In particular, the critical point of the nonlinear system is of the same type and stability as the critical point of the linearization *except* in two special cases:

(i) If the linearization has a center (complex conjugate eigenvalues with zero real part), the nonlinear system can have a center or a stable or unstable spiral point.

(ii) If the linearization has a node with equal real eigenvalues, the nonlinear system can have a node or a spiral point, but with the same stability as the node in the linearization.

(f) You should be able to sketch and interpret phase planes (graphs of y versus x) and graphs of x and y versus t.

4. **Application: interacting species.** (Section 6.3.)

   (a) **The predator-prey model.**

   $$x' = ax - pxy,$$
   $$y' = -by + qxy,$$

   where x is the prey population, y is the predator population, and a, b, p, q are positive constants. This has a nonzero critical point at $(b/q, a/p)$, and solutions orbit it stably with angular frequency $\sqrt{ab}$.

   (b) **The competing species model.**

   $$x' = a_1x - b_1x^2 - c_1xy,$$
   $$y' = a_2y - b_2y^2 - c_2xy,$$

   where x and y are the populations of the two species and the other numbers are positive constants. The two populations grow logistically on their own but also compete over resources, leading to nesgative effects of their interaction. The system has four critical points: one at the origin, two of the form $(K_x, 0)$ and $(0, K_y)$ at which one species is extinct and the other is at carrying capacity, and a fourth where both species have nonzero population. The fourth critical point is an unstable saddle point if $c_1c_2 > b_1b_2$, and an asymptotically stable node if $c_1c_2 = b_1b_2$. In the unstable case, which species survives and which goes extinct depends on the species' initial values.

   (c) **Other examples.** You should be able to apply this sort of reasoning to other situations. What if the species in (b) cooperate (so that the negative terms $-c_1xy$, $-c_2xy$ are replaced by positive ones)? What if the logistic terms in (b) are added into the predator-prey model of (a)? What if one species is a scavenger that reproduces not based on the other species' population, but rather its death rate? What if there are more than two species forming a food chain or food network?