# Math 303, Practice Final

**Instructions:** *The real exam is 2 hour long and has 15 questions, worth 200 points in total. No calculators or notes will be permitted.*

*If you want your work graded, make sure it's understandable and it's clear which question it's referring to. If you tear pages out, write your name on top of them. If you finish early, you can hand the exam in up front and leave.*

Name: ______________________________________________

Section (circle one): 9-10:15 10:30-11:45

v1.1

**Part I: Multiple Choice**

*Each question is worth 5 points, and has a single correct answer. There will be no partial credit.*

1. A mass of 0.2 kg is attached to a spring with spring constant 0.8 kg/s$^2$ and damping constant 0.1 kg/s. Which of the following forcing functions will provoke the largest response in the spring?

   (a) $F(t) = \sin(0.7t)$

   (b) $F(t) = \sin(1.7t)$

   (c) $F(t) = \cos(3.7t)$

   (d) $F(t) = \cos(5(t-1))$

2. Which one of the following is an example of a regular Sturm-Liouville problem?

(a)

$$\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right) - xy + \lambda y = 0,\ -1 < x < 1;\ y(-1) = y'(1) = 0$$

(b)

$$X'' + \lambda X = 0,\ 0 < x < 5$$

(c)

$$y'' + (1+\lambda)y = 0,\ 0 < x < 2;\ y(0) = y'(0) = 0$$

(d)

$$U'' + x^2U + \lambda U = 0,\ 0 < x < 1;\ y(0) = 3y(1) - y'(1) = 0$$

(e)

$$r^2U'' + rU' + U = 0,\ 0 < r < 3;\ U(0) = U'(3) = 0$$

3. Suppose that $f$ is a piecewise smooth, $2\pi$-periodic function with Fourier series

$$f(t) \sim \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx).$$

Which of the following is the Fourier series of $f'(t)$?

(a) $\sum_{n=1}^{\infty} \sin(nx)$

(b) $\sum_{n=1}^{\infty} \cos(nx)$

(c) $\sum_{n=1}^{\infty} - \cos(nx)$

(d) $\sum_{n=1}^{\infty} \sin(2nx)$

(e) None of the above / impossible to say without further information.

4. Consider the regular Sturm-Liouville problem

$$X'' + \lambda x X = 0,\ 1 \le x \le 2;\ X'(1) = X'(2) = 0.$$

Suppose that $\lambda_n$ are the eigenvalues for this problem, $X_n$ are the associated eigenfunctions, and $y$ is an arbitrary function on $[1, 2]$, with eigenfunction series

$$y(x) \sim \sum C_n X_n(x).$$

What is the correct expression for $C_n$?

(a)
$$C_n = \int_1^2 y X_n \, dx$$

(b)
$$C_n = \frac{\int_1^2 y X_n \, dx}{\int_1^2 X_n^2 \, dx}$$

(c)
$$C_n = \frac{\int_1^2 x y X_n \, dx}{\int_1^2 x X_n^2 \, dx}$$

(d) $C_n = 1$ if $y = X_n$ and 0 otherwise.

(e) None of the above.

5. One possible steady-state heat distribution on a disk of radius 1 is

$$u(r,\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{\pi n^2} r^n \cos(n\theta).$$

Which of the following boundary conditions does this satisfy?

(a) $u(1,\theta) = |\theta|$ (for $-\pi \leq \theta \leq \pi$)

(b) $u(1,\theta) = \theta$ (for $0 \leq \theta \leq 2\pi$)

(c) $u(1,\theta) = \theta$ (for $-\pi \leq \theta \leq \pi$)

(d) $u(1,\theta) = \theta^2$ (for $-\pi \leq \theta \leq \pi$)

(e) $u(1,\theta) = u(0,\theta) = 0$

6. Which of the following problems, consisting of a partial differential equation with some boundary conditions, satisfies the following property?

   (P) If $u_1$ and $u_2$ are solutions to the problem, then so is any linear combination $C_1u_1 + C_2u_2$.

   (a) $u_{xx} + u_{yy} = 0$ for $0 \leq x \leq 1$, $0 \leq y \leq 2$; $u(0, y) = u(1, y) = 0$, $u(x, 0) = x$, $u(x, 1) = u_y(x, 1)$.

   (b) $r^2u_{rr} + ru_r + u_{\theta\theta} = 0$ for $0 \leq r \leq 5$; $u(r, \theta) = u(r, \theta + 2\pi)$; $u_r(5, \theta) = 0$.

   (c) $u_{tt} = v^2u_{xx}$ for $0 \leq x \leq 3$; $u(0, t) = u(L, t) = 0$, $u_t(x, 0) = 0$, $u(x, 0) = x(L - x)$.

   (d) $u_t - Ku_{xx} = 0$ for $0 \leq x \leq L$; $u(0, t) = u_x(L, t) - u(L, t)^2 = 0$.

7. When a certain string of length 1 is allowed to vibrate with its ends fixed, it vibrates with fundamental frequency 330 Hz. As the string vibrates, the point 3/4 of the way along its length is also held fixed. What (to the nearest hertz) is the new fundamental frequency of the resulting vibrations?

   (a) 83 Hz
   (b) 248 Hz
   (c) 440 Hz
   (d) 660 Hz
   (e) 1320 Hz

8. A population of predators feeds on a population of prey animals, leading to a decrease in the prey population, and an increase in the predator population, both of which are proportional to the product of the predator and prey populations. If the predator population is left alone, it decays exponentially, and if the prey population is left alone, it grows logistically. Which of the following systems of differential equations is the best model for this situation? Assume all constants are positive.

   (a) $x' = a_1 x - b_1 x^2 - c_1 xy, \quad y' = -a_2 y + c_2 xy$
   (b) $x' = a_1 x + b_1 x^2 - c_1 xy, \quad y' = -a_2 y + b_2 y^2 - c_2 xy$
   (c) $x' = a_1 x - c_1 xy - d_1 x^2 y, \quad y' = \exp(-a_2 y) + c_2 xy$
   (d) $x' = a_1 x - c_1 y, \quad y' = -a_2 y + c_2 xy$
   (e) $x' = a_1 x + b_1 y - c_1 xy, \quad y' = -a_2 y + b_2 x - c_2 xy$

9. Which of the following functions is piecewise smooth on its domain?

   (a) The 2-periodic function on the real line, equal to $|x|$ on the interval $[-1, 1]$.

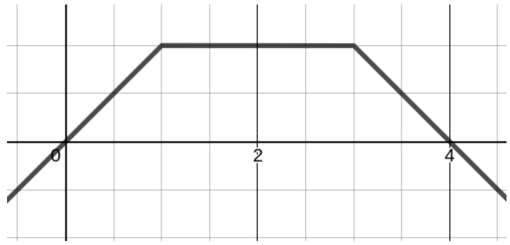
   (b) The function $\ln(x)$ on the interval $[1, 2]$.

   (c) The constant function 0 on the interval $[0, 10]$.

   (d) The 1-periodic function on the real line, equal to $\sin(x) + 2\cos(x)$ on $[0, 1]$.

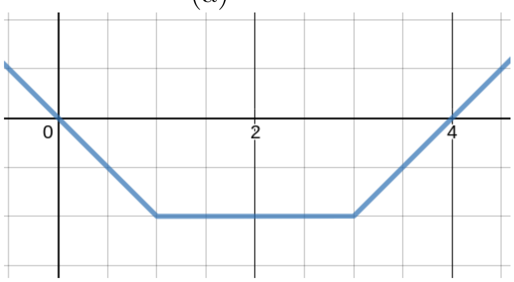
   (e) All of the above.

10. This is the position of a string of length 4 and wave speed 2 at $t = 0$:

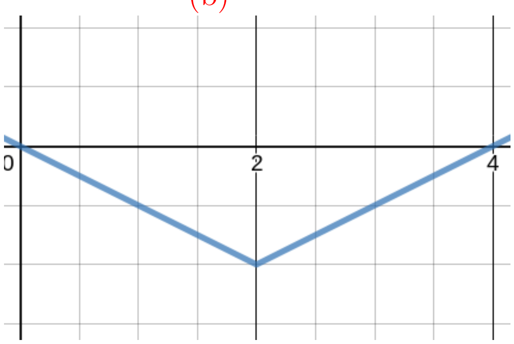



The string's ends are fixed, and it has no initial velocity. What is the position of the string at time $t = 2$?

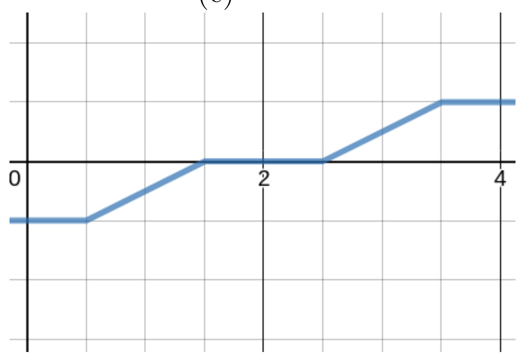
(a)



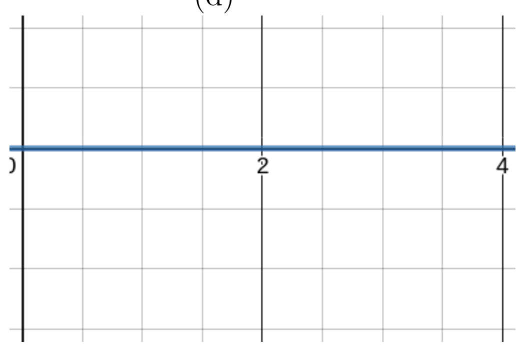

(b)




(c)




(d)




(i. e. the zero function)

**Part II: Long Answer Problems**

*Each of these questions is worth 30 points. You can get partial credit on these, based on the work you do towards an answer. To get the most partial credit, make sure your work is legible and understandable.*

11. Find all the critical points of the following nonlinear system, and describe the type and stability of each critical point as fully as you can.

$$x' = y^2 - 1,$$
$$y' = \sin(x) - y.$$

12. Find the general solution to the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}.$$

13. A metal rod of thermal diffusivity $K$ and length $L$ is laterally insulated, so that the temperature distribution $u(x, t)$ satisfies the heat equation

$$u_t = K u_{xx}.$$

The 0 end of the rod undergoes heat transfer with a surrounding medium at temperature zero, while the $L$ end of the rod is insulated, so $u$ satisfies the boundary conditions

$$hu(0, t) - u_x(0, t) = 0, \quad u_x(L, t) = 0.$$

Find the general solution $u(x, t)$.

14. An undamped spring-mass system of mass 1 kg and spring constant 2 kg/s$^2$ is forced by a sawtooth wave function $F(x)$, which is 2-periodic and given by $F(t) = t$ on the interval $[-1, 1]$. Give a formula for the displacement of the mass, $x(t)$, as a function of time.

15. An infinite metal strip of the form $\{(x, y) : 0 \leq x \leq 1,\ 0 \leq y\}$ is delivered to you in a mysterious, infinitely long package, together with the following instructions:

HOLD X AXIS EDGE AT 100 DEGREES
HOLD OTHER TWO EDGES AT 0 DEGREES
MAKE SURE TEMPERATURE IS BOUNDED AND CONTINUOUS
WAIT FOR STEADY STATE
DONT THINK ABOUT THE CORNERS

What is the temperature distribution $u(x, y)$ after you follow the instructions?