Math 303, Practice Final Solutions

Instructions: *The real exam is 2 hour long and has 15 questions, worth 200 points in total. No calculators or notes will be permitted.*

If you want your work graded, make sure it's understandable and it's clear which question it's referring to. If you tear pages out, write your name on top of them. If you finish early, you can hand the exam in up front and leave.

Name: ____

Section (circle one): 9-10:15 10:30-11:45

v1.1

Part I: Multiple Choice

Each question is worth 5 points, and has a single correct answer. There will be no partial credit.

1. A mass of 0.2 kg is attached to a spring with spring constant 0.8 kg/s^2 and damping constant 0.1 kg/s. Which of the following forcing functions will provoke the largest response in the spring?

   (a) $F(t) = \sin(0.7t)$

   (b) $F(t) = \sin(1.7t)$

   (c) $F(t) = \cos(3.7t)$

   (d) $F(t) = \cos(5(t-1))$

   Answer: (b). All the forcing functions have the same amplitude, so the one with the largest response amplitude will be the one closest to the resonant frequency of the spring, which is the frequency of its undamped oscillations. We can find the undamped oscillations explicitly:

$$0.2x'' + 0.1x' + 0.8x = 0$$

   is a constant-coefficient ODE with characteristic polynomial

$$0.2r^2 + 0.1r + 0.8,$$

   which has roots

$$r = \frac{-1 \pm \sqrt{-63}}{4} = \frac{-1}{4} \pm i\frac{\sqrt{63}}{4}$$

   So the unforced solutions are

$$x = C_1 e^{-t/4} \cos(\sqrt{63}t/4) + C_2 e^{-t/4} \sin(\sqrt{63}t/4).$$

   These have angular frequency $\sqrt{63}/4 \approx 2$. (b) is the only forcing function with a frequency reasonably close to this.

2. Which one of the following is an example of a regular Sturm-Liouville problem?

(a)
$$\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right) - xy + \lambda y = 0,\ -1 < x < 1;\ y(-1) = y'(1) = 0$$

(b)
$$X'' + \lambda X = 0,\ 0 < x < 5$$

(c)
$$y'' + (1+\lambda)y = 0,\ 0 < x < 2;\ y(0) = y'(0) = 0$$

(d)
$$U'' + x^2U + \lambda U = 0,\ 0 < x < 1;\ y(0) = 3y(1) - y'(1) = 0$$

(e)
$$r^2U'' + rU' + U = 0,\ 0 < r < 3;\ U(0) = U'(3) = 0$$

Answer: (d). (a) isn't regular since the coefficient function $1/x$ is discontinuous at 0, which is in the interval of definition. (b) isn't regular since it has no boundary conditions. (c) isn't regular since it has two boundary conditions at one endpoint of the interval, and no boundary condition at the other end. (e) isn't regular since the differential equation doesn't have a λ in it, meaning there's no eigenvalue to find.

3. Suppose that f is a piecewise smooth, 2π-periodic function with Fourier series

$$f(t) \sim \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx).$$

Which of the following is the Fourier series of $f'(t)$?

(a) $\sum_{n=1}^{\infty} \sin(nx)$
(b) $\sum_{n=1}^{\infty} \cos(nx)$
(c) $\sum_{n=1}^{\infty} -\cos(nx)$
(d) $\sum_{n=1}^{\infty} \sin(2nx)$
(e) None of the above / impossible to say without further information.

Answer: (e). The temptation is to differentiate term by term, giving (b), but we can't do this unless we know that f is continuous, which we do not. The fact that (b) diverges at $x = 0$ should be at least a source of worry. In fact, f is a discontinuous sawtooth wave, whose derivative is constant wherever it's defined, so (b) is the wrong answer.

4. Consider the regular Sturm-Liouville problem

$$X'' + \lambda x X = 0,\ 1 \le x \le 2;\ X'(1) = X'(2) = 0.$$

Suppose that λ_n are the eigenvalues for this problem, X_n are the associated eigenfunctions, and y is an arbitrary function on $[1, 2]$, with eigenfunction series

$$y(x) \sim \sum C_n X_n(x).$$

What is the correct expression for C_n?

(a)
$$C_n = \int_1^2 y X_n \, dx$$

(b)
$$C_n = \frac{\int_1^2 y X_n \, dx}{\int_1^2 X_n^2 \, dx}$$

(c)
$$C_n = \frac{\int_1^2 x y X_n \, dx}{\int_1^2 x X_n^2 \, dx}$$

(d) $C_n = 1$ if $y = X_n$ and 0 otherwise.

(e) None of the above.

Answer: (c). This is an application of the formula for the coefficients of an eigenfunction series, which is in section 10.1 of the book.

5. One possible steady-state heat distribution on a disk of radius 1 is

$$u(r,\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{\pi n^2} r^n \cos(n\theta).$$

Which of the following boundary conditions does this satisfy?

(a) $u(1,\theta) = |\theta|$ (for $-\pi \leq \theta \leq \pi$)
(b) $u(1,\theta) = \theta$ (for $0 \leq \theta \leq 2\pi$)
(c) $u(1,\theta) = \theta$ (for $-\pi \leq \theta \leq \pi$)
(d) $u(1,\theta) = \theta^2$ (for $-\pi \leq \theta \leq \pi$)
(e) $u(1,\theta) = u(0,\theta) = 0$

Answer: (a) We have

$$u(1,\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi)) - 1)}{\pi n^2} \cos(n\theta),$$

and the question is which of the functions given has this as its Fourier series. (Sanity check: they are all functions on the circle, so they're 2π-periodic functions of θ, which means they have Fourier series that are sums of $\sin(n\theta)$ and $\cos(n\theta)$ for various n.) It's clearly not (e) ((e) also specifies a value at $r = 0$ which doesn't agree with the given function). Also, our Fourier series is a cosine series, so it converges to an even function, so we can rule out (b) and (c). We can choose between (a) and (d) by computing their Fourier series. For (a), we have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi} \pi|\theta|\, d\theta = \frac{1}{\pi}\left(\int_0^{\pi} \theta\, d\theta - \int_{-\pi}^{0} \theta\, d\theta\right) \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2} + \frac{\pi^2}{2}\right) = \pi. \end{aligned}$$

For (d), we have

$$a_0 = \frac{1}{\pi} \int_{-\pi} \pi\theta^2\, d\theta = \frac{1}{\pi} \cdot \frac{2\pi^3}{3} = \frac{2\pi^2}{3}.$$

Since our $a_0/2$ is $\pi/2$, (a) is the only possible answer (and we could check it by computing the rest of its Fourier coefficients.)

6. Which of the following problems, consisting of a partial differential equation with some boundary conditions, satisfies the following property?

(P) If u_1 and u_2 are solutions to the problem, then so is any linear combination $C_1 u_1 + C_2 u_2$.

(a) $u_{xx} + u_{yy} = 0$ for $0 \leq x \leq 1$, $0 \leq y \leq 2$; $u(0, y) = u(1, y) = 0$, $u(x, 0) = x$, $u(x, 1) = u_y(x, 1)$.

(b) $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ for $0 \leq r \leq 5$; $u(r, \theta) = u(r, \theta + 2\pi)$; $u_r(5, \theta) = 0$.

(c) $u_{tt} = v^2 u_{xx}$ for $0 \leq x \leq 3$; $u(0, t) = u(L, t) = 0$, $u_t(x, 0) = 0$, $u(x, 0) = x(L - x)$.

(d) $u_t - K u_{xx} = 0$ for $0 \leq x \leq L$; $u(0, t) = u_x(L, t) - u(L, t)^2 = 0$.

Answer: (b). For instance, suppose u_1 and u_2 are solutions to (a), and let $u = C_1 u_1 + C_2 u_2$ for some numbers C_1 and C_2. Then

$$u(x, 0) = C_1 u_1(x, 0) + C_2 u_2(x, 0) = C_1 x + C_2 x.$$

If $C_1 + C_2 = 1$, this is not equal to x. Similarly, in (c), the condition $u(x, 0) = x(L - x)$ fails (P), and in (d), $u_x(L, t) - u(L, t)^2 = 0$ fails (P). That leaves (b). By the way, (b) has a physical interpretation you might have recognized: it describes steady-state heat on a disk of radius 5 which is insulated around its boundary.

7. When a certain string of length 1 is allowed to vibrate with its ends fixed, it vibrates with fundamental frequency 330 Hz. As the string vibrates, the point 3/4 of the way along its length is also held fixed. What (to the nearest hertz) is the new fundamental frequency of the resulting vibrations?

   (a) 83 Hz
   (b) 248 Hz
   (c) 440 Hz
   (d) 660 Hz
   (e) 1320 Hz

   Answer: (e). The Fourier series solution to the wave equation tells us that every vibration of this string with $L = 1$ m is a sum of standing waves of the form

   $$\sin(n\pi x)\cos(n\pi vt), \quad \sin(n\pi x)\sin(n\pi vt).$$

   The angular frequency of such waves is $n\pi v$ (radians per second), so their frequency in Hertz (i. e. cycles per second) is $nv/2$. The lowest such frequency is $v/2$, meaning that v is 660 m/s. Now, if $x = 3/4$ is held fixed, the only standing waves that can survive are those with $y(3/4, t) = 0$ for all t. This means

   $$\sin(3n\pi/4) = 0,$$

   which means n is a multiple of 4. So the new lowest angular frequency is $4\pi v$, and the new lowest frequency in Hertz is $2v = 1320$ Hz.

   Three notes: first, you don't need to calculate v, or even know the length of the string, to do this – you could just recognize, say by drawing graphs, that fixing this point causes the wavelength of the longest-wavelength standing wave to shrink by 1/4, and thus its frequency to grow by a factor of 4. Second, the situation is different if the entire portion of the string from $x = 3/4$ to $x = 1$ is fixed. In this case, the string would start behaving like a string of length 3/4, and since v depends on the string's physical properties and presumably wouldn't change, the new fundamental frequency would be 4/3 times the old one, or 440 Hz. Third, the argument above doesn't work if you fix a point with *irrational* x-value. In this case, none of the old standing waves could survive. I'm not sure what happens, but I'd guess that you would just completely silence the string.

8. A population of predators feeds on a population of prey animals, leading to a decrease in the prey population, and an increase in the predator population, both of which are proportional to the product of the predator and prey populations. If the predator population is left alone, it decays exponentially, and if the prey population is left alone, it grows logistically. Which of the following systems of differential equations is the best model for this situation? Assume all constants are positive.

   (a) $x' = a_1x - b_1x^2 - c_1xy, \quad y' = -a_2y + c_2xy$

   (b) $x' = a_1x + b_1x^2 - c_1xy, \quad y' = -a_2y + b_2y^2 - c_2xy$

   (c) $x' = a_1x - c_1xy - d_1x^2y, \quad y' = \exp(-a_2y) + c_2xy$

   (d) $x' = a_1x - c_1y, \quad y' = -a_2y + c_2xy$

   (e) $x' = a_1x + b_1y - c_1xy, \quad y' = -a_2y + b_2x - c_2xy$

   Answer: (a). Say x is the prey and y is the predators. The first sentence tells us that x' has a term of the form $-Axy$ and y has a term of the form $+Bxy$, where I'm using capital letters for various positve constants. The second sentence tells us that y' has a term like $-Cy$ (which creates exponential decay); and that x' has terms like $Dx - Ex^2$ (which creates logistic growth). If you have trouble remembering this stuff, it's good to just logic through what each term means – for example, a term Dx in x' means x has a tendency to grow proportional to the current population (that is, exponentially), and a term $-Ex^2$ means that this growth slows as x gets larger. Anyway, (a) is the only answer of this form. We should also make sure that none of the other answers make sense if x is the predators and y is the prey instead, but they don't.

9. Which of the following functions is piecewise smooth on its domain?

   (a) The 2-periodic function on the real line, equal to $|x|$ on the interval $[-1, 1]$.

   (b) The function $\ln(x)$ on the interval $[1, 2]$.

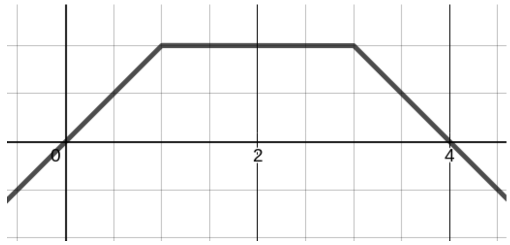
   (c) The constant function 0 on the interval $[0, 10]$.

   (d) The 1-periodic function on the real line, equal to $\sin(x) + 2\cos(x)$ on $[0, 1]$.

   (e) All of the above.

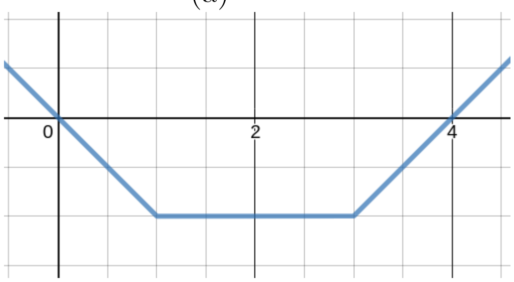
   Answer: (e). "Piecewise smooth" means we can break the domain into intervals such that f is continuous on each interval; it has finite, well-defined one-sided limits at each endpoint of each interval; and the same conditions apply to f'. Now, (c) is continuously differentiable on its whole domain. So is (b) – $\ln(x)$ goes to $-\infty$ as x goes to 0, but this isn't in the domain. (a) is continuous everywhere (draw a graph) and differentiable everywhere except $x =$ an integer. At these points, the one-sided limits of $f'(x)$ are ± 1. Likewise, (d) is continuously differentiable on each interval $[n, n+1]$, and doesn't do anything ridiculous at the discontinuities.

10. This is the position of a string of length 4 and wave speed 2 at $t = 0$:

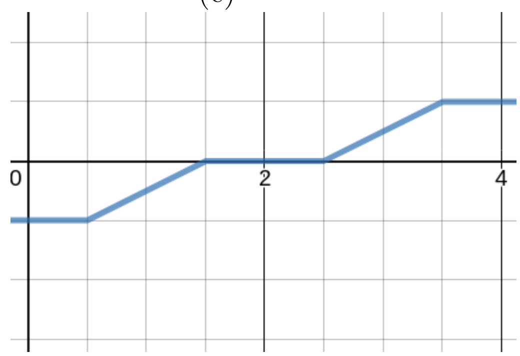


The string's ends are fixed, and it has no initial velocity. What is the position of the string at time $t = 2$?

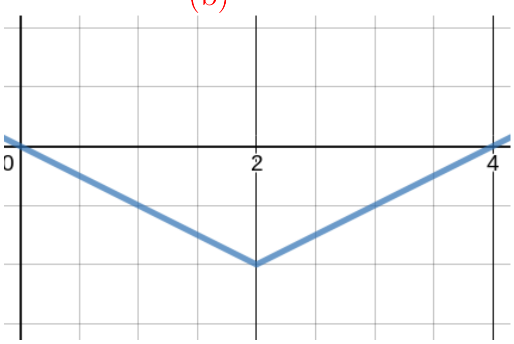
(a)



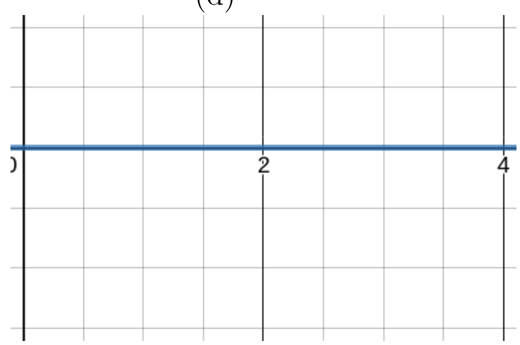
(c)



(b)



(d)



(i. e. the zero function)

Answer: (a). The trick here is the d'Alembert solution,

$$y(x,t) = \frac{1}{2}\left(F_{\text{odd}}(x+vt) \pm F_{\text{odd}}(x-vt)\right),$$

where y has zero initial velocity, $f(x) = y(x,0)$ is the initial position function, and F_{odd} is the odd $2L$-periodic extension of f. In practice, you can draw F_{odd} from f, translate it to the left and right, and take the average, all by hand (and you should do this so you understand!). In this case, we want to take the average of $F_{\text{odd}}(x+4)$ and $F_{\text{odd}}(x-4)$. It turns out that both of these are the same as the graph of (a), so their average is also (a).

Part II: Long Answer Problems

Each of these questions is worth 30 points. You can get partial credit on these, based on the work you do towards an answer. To get the most partial credit, make sure your work is legible and understandable.

11. Find all the critical points of the following nonlinear system, and describe the type and stability of each critical point as fully as you can.

$$x' = y^2 - 1,$$
$$y' = \sin(x) - y.$$

The critical points are the solutions to

$$0 = y^2 - 1,$$
$$0 = \sin(x) - y.$$

These are the points $(\pi/2 + 2k\pi, 1)$ and $(3\pi/2 + 2k\pi, -1)$ where k is any integer. The Jacobian of the system is

$$J(x, y) = \begin{pmatrix} 0 & 2y \\ \cos(x) & -1 \end{pmatrix}.$$

At these critical points,

$$J = \begin{pmatrix} 0 & \pm 2 \\ 0 & -1 \end{pmatrix}.$$

The determinant of J is zero, so even though the critical points are isolated, the system is not "almost linear". That is, we can't conclude anything about the system's behavior near its critical points from the linearizations there.

You can graph this system on `pplane` and see that its critical points are actually of a type we didn't study. They have one tangent direction where solutions approach the critical point, and another one where they approach it from one side and leave it from the other side.

This is a badly written question, but you can easily write more by yourself. Just pick two functions $F(x, y)$ and $G(x, y)$, ask yourself the same question about the system $x' = F(x, y)$ and $y' = G(x, y)$, and check your answers in `pplane`. There are lots of simple systems in the exercises to chapter 6 of the book you can also use.

12. Find the general solution to the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}.$$

The matrix is upper triangular, so its eigenvalues are 1, 1, and 3. For $\lambda = 3$, we find the eigenvector $(1, 1, 1)^T$, giving a solution

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t}.$$

For $\lambda = 1$, we have the eigenvector $(1, 0, 0)^T$, giving another solution

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t}.$$

However, there is only a one-dimensional space of eigenvectors for $\lambda = 1$. To find our third independent solution, we need to look for a generalized eigenvector $\mathbf{w}$ satisfying

$$(A - 1 \cdot I)\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

One such vector is $(0, 1, 0)^T$. So we get a third solution,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t}.$$

The general solution is then

$$\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + C_3 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} \right).$$

It's very likely you got something that looks different, in which case it's good practice to check that it describes the same set of solutions as my solution above.

13. A metal rod of thermal diffusivity K and length L is laterally insulated, so that the temperature distribution $u(x,t)$ satisfies the heat equation

$$u_t = K u_{xx}.$$

The 0 end of the rod undergoes heat transfer with a surrounding medium at temperature zero, while the L end of the rod is insulated, so u satisfies the boundary conditions

$$hu(0,t) - u_x(0,t) = 0, \quad u_x(L,t) = 0.$$

Find the general solution $u(x,t)$.

We separate variables,

$$u(x,t) = X(x)T(t).$$

Then the differential equation becomes

$$XT' = KX''T$$

or

$$\frac{X''}{X} = \frac{T'}{KT}.$$

Each side is independent of one of the two variables, so they're both equal to the same constant, say $-\lambda$. The boundary conditions also tell us that

$$hX(0) - X'(0) = 0, \quad X'(L) = 0.$$

Together with the X equation, these give a regular Sturm-Liouville problem for X, namely

$$X'' + \lambda X = 0 \quad (0 < X < L); hX(0) - X'(0) = 0, \quad X'(L) = 0.$$

The problem is nonnegative (assuming that h is a *positive* constant), so the eigenvalues are nonnegative. If $\lambda = 0$, then X is a linear function of x,

$$X = C_1 + C_2 x.$$

You can check that the only function like this satisfying the boundary conditions is the zero function. If $\lambda > 0$, then

$$X = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

The boundary conditions imply that

$$\begin{aligned} 0 &= hX(0) - X'(0) = hC_1 - \sqrt{\lambda}C_2, \\ 0 &= X'(L) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L). \end{aligned}$$

From the first equation, we get $C_1 = \frac{\sqrt{\lambda}}{h}C_2$. The second is then

$$0 = -\frac{\lambda}{h}C_2 \sin(\sqrt{\lambda}L) + \sqrt{\lambda}C_2\cos(\sqrt{\lambda}L).$$

Dividing by C_2 and rearranging, we get

$$\frac{\sqrt{\lambda}}{h} = \frac{\cos(\sqrt{\lambda}L)}{\sin(\sqrt{\lambda}L)} = \cot(\sqrt{\lambda}L).$$

In other words, $\sqrt{\lambda}L$ satisfies the equation

$$\frac{x}{hL} = \cot(x).$$

In other words, if β_n is the nth positive solution to this equation, then $\beta_n = \sqrt{\lambda_n}L$, so $\lambda_n = \beta_n^2/L^2$. We check that these numbers actually exist by sketching a graph of the cotangent function. In sum, we've found the eigenvalues,

$$\lambda_n = \frac{\beta_n^2}{L^2}, \quad \beta_n = n\text{th positive solution to } x/hL = \cot(x),$$

and the eigenfunctions

$$X_n = \frac{\sqrt{\lambda_n}}{h}\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x) = \frac{\beta_n}{hL}\cos\left(\frac{\beta_n x}{L}\right) + \sin\left(\frac{\beta_n x}{L}\right).$$

For each λ_n, we have a first-order equation for the corresponding function T_n, namely

$$T_n' = -\lambda_n K T_n.$$

So

$$T_n = \exp(-K\lambda_n t) = \exp\left(\frac{-K\beta_n^2}{L^2}t\right).$$

Thus the general solution is

$$u = \sum C_n X_n T - n = \sum_{n=1}^{\infty} C_n \left[\frac{\beta_n}{hL}\cos\left(\frac{\beta_n x}{L}\right) + \sin\left(\frac{\beta_n x}{L}\right)\right]\exp\left(\frac{-K\beta_n^2}{L^2}t\right),$$

where β_n is the nth positive solution to $x/hL = \cot(x)$.

I hope I don't have to say this, but if you solve a problem like this on the exam and you want full credit, you do need to define β_n in words at some point in your solution!

14. An undamped spring-mass system of mass 1 kg and spring constant 2 kg/s^2 is forced by a sawtooth wave function $F(x)$, which is 2-periodic and given by $F(t) = t$ on the interval $[-1, 1]$. Give a formula for the displacement of the mass, $x(t)$, as a function of time.

We want to solve the differential equation

$$x'' + 2x = F(t).$$

We start by writing F as a Fourier series. It is odd, so this will be a sine series. We have

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 t\sin(n\pi t)\,dt = 2\left[-t\frac{\cos(n\pi t)}{n\pi}\right]_0^1 + 2\int_0^1 \frac{\cos(n\pi t)}{n\pi}\,dt \\ &= \frac{-2\cos(n\pi)}{n\pi} + 2\left[\frac{\sin(n\pi)}{n^2\pi^2}\right]_0^1 = \frac{-2\cos(n\pi)}{n\pi}. \end{aligned}$$

So we have to solve

$$x'' + 2x = \sum_{n=1}^{\infty} \frac{-2\cos(n\pi)}{n\pi}\sin(n\pi t).$$

Let's first solve

$$x_n'' + 2x_n = \sin(n\pi t).$$

We do this with undetermined coefficients, $x_n = A_n \sin(n\pi t)$. Then

$$-n^2\pi^2 A_n \sin(n\pi t) + 2A_n \sin(n\pi t) = \sin(n\pi t).$$

Comparing coefficients gives

$$A_n(-n^2\pi^2 + 2) = 1,$$

so

$$A_n = \frac{1}{2 - n^2\pi^2}.$$

(Note that 2 isn't a square, so we're never dividing by zero here. What would happen if the spring constant was 4 instead?) Thus,

$$x_n = \frac{\sin(n\pi t)}{2 - n^2\pi^2}.$$

The formula for the steady periodic solution is then

$$x = \sum b_n x_n = \sum_{n=1}^{\infty} \frac{-2\cos(n\pi)}{n\pi} \cdot \frac{\sin(n\pi t)}{2 - n^2\pi^2}.$$

I forgot to specify whether the steady periodic solution or the general solution was required in this problem. The general solution is given by adding the general solution to the associated homogeneous equation, $x'' + 2x = 0$, to this particular solution. It is

$$x = \sum_{n=1}^{\infty} \frac{-2\cos(n\pi)}{n\pi} \cdot \frac{\sin(n\pi t)}{2 - n^2} + C_1\cos(\sqrt{2}t) + C_2\sin(\sqrt{2}t).$$

15. An infinite metal strip of the form $\{(x,y) : 0 \le x \le 1,\ 0 \le y\}$ is delivered to you in a mysterious, infinitely long package, together with the following instructions:

HOLD X AXIS EDGE AT 100 DEGREES
HOLD OTHER TWO EDGES AT 0 DEGREES
MAKE SURE TEMPERATURE IS BOUNDED AND CONTINUOUS
WAIT FOR STEADY STATE
DONT THINK ABOUT THE CORNERS

What is the temperature distribution $u(x,y)$ after you follow the instructions?

Since we want the steady-state temperature distribution, we are looking for solutions to

$$u_{xx} + u_{yy} = 0$$

on the given domain, subject to the boundary conditions

$$u(0,y) = u(1,y) = 0 \quad (0 < y),$$

$$u(x,0) = 100 \quad (0 < x < 1),$$

$$u \text{ is bounded as } y \to +\infty.$$

Let's ignore the nonhomogeneous boundary condition $u(x,0) = 100$ at first. Separate variables, writing

$$u = X(x)Y(y),$$

and obtaining

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

We start with the equation for X, as there are two boundary conditions affecting it. Namely, the conditions on the sides of the strip imply that

$$X(0) = X(1) = 0.$$

Thus we want to solve

$$X'' + \lambda X = 0,\ (0 < x < 1), \quad X(0) = X(1) = 0,$$

which is a regular, nonnegative Sturm-Liouville problem. Since it's nonnegative, we can ignore the case where λ is negative, and it is easily checked that the $\lambda = 0$ case has no nontrivial solutions. If $\lambda > 0$, then

$$X = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x),$$

and the boundary conditions imply

$$C_1 = 0, \quad C_1 \cos(\sqrt{\lambda}) + C_2 \sin(\sqrt{\lambda}) = 0.$$

Thus $\sin(\sqrt{\lambda})$ is zero, so $\sqrt{\lambda}$ is an integer multiple of π. Let

$$\lambda_n = n^2\pi^2, \quad n = 1, 2, 3, \dots .$$

An associated eigenfunction is

$$X_n = \sin(n\pi x).$$

Now we turn to Y. For each n, we have the problem

$$Y_n'' - \lambda_n Y_n = 0, \quad (0 < y)$$

where Y_n is supposed to be bounded as $y \to \infty$. The solutions to the ODE are

$$Y_n = C_1 \exp(-\sqrt{\lambda_n} y) + C_2 \exp(\sqrt{\lambda_n} y) = C_1 e^{-n\pi y} + C_2 e^{n\pi y}.$$

This is only bounded if $C_2 = 0$; otherwise, it grows to $\pm\infty$ (depending on the sign of C_2) as $y \to +\infty$. So let

$$Y_n = e^{n\pi y}.$$

Then for each n,

$$u_n = X_n Y_n = \sin(n\pi x) e^{n\pi y}$$

is a building block solution to the problem. The general solution is

$$u = \sum_{n=1}^{\infty} C_n \sin(n\pi x) e^{n\pi y}.$$

Finally, we need the particular solution with

$$u(x, 0) = 100 \quad (0 < x < 1).$$

So

$$100 = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \quad (0 < x < 1).$$

In other words, $\sum C_n \sin(n\pi x)$ is the Fourier sine series for the odd, 2-periodic extension of the constant function 100. We calculate

$$C_n = 2\int_0^1 100 \sin(n\pi x) = 200 \left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{200(1 - \cos(n\pi))}{n\pi}.$$

So the particular solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{200(1 - \cos(n\pi)}{n\pi} \sin(n\pi x) e^{n\pi y}.$$