Math 303, Practice midterm

Instructions: *The real exam is 1 hour long. No calculators and notes will be permitted. Partial credit will be given on the long-answer questions, based on the work you do towards the answer.*

1. Consider a linear system of the form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where A is a 3×3 constant matrix. Which of the following statements is true?

   (a) The system has three solutions.

   (b) The system has at most three solutions, but may have fewer if A has a repeated eigenvalue.

   (c) The system has infinitely many solutions.

   (d) The number of solutions depends on the Wronskian.

   (e) We cannot know how many solutions the system has without knowing more about A.

2. Consider an almost linear system of the form

$$x' = F(x, y), \quad y' = G(x, y).$$

Suppose that this system has an unstable spiral point at $(0, 0)$. Let J be the Jacobian of the system at $(0, 0)$. Which of the following can describe the eigenvalues of J? *Circle all that apply.*

   (a) Two distinct positive real eigenvalues

   (b) Two distinct negative real eigenvalues

   (c) A positive and a negative real eigenvalue

   (d) A repeated positive real eigenvalue

   (e) A repeated negative real eigenvalue

   (f) Two complex eigenvalues with zero real part

   (g) Two complex eigenvalues with positive real part

   (h) Two complex eigenvalues with negative real part

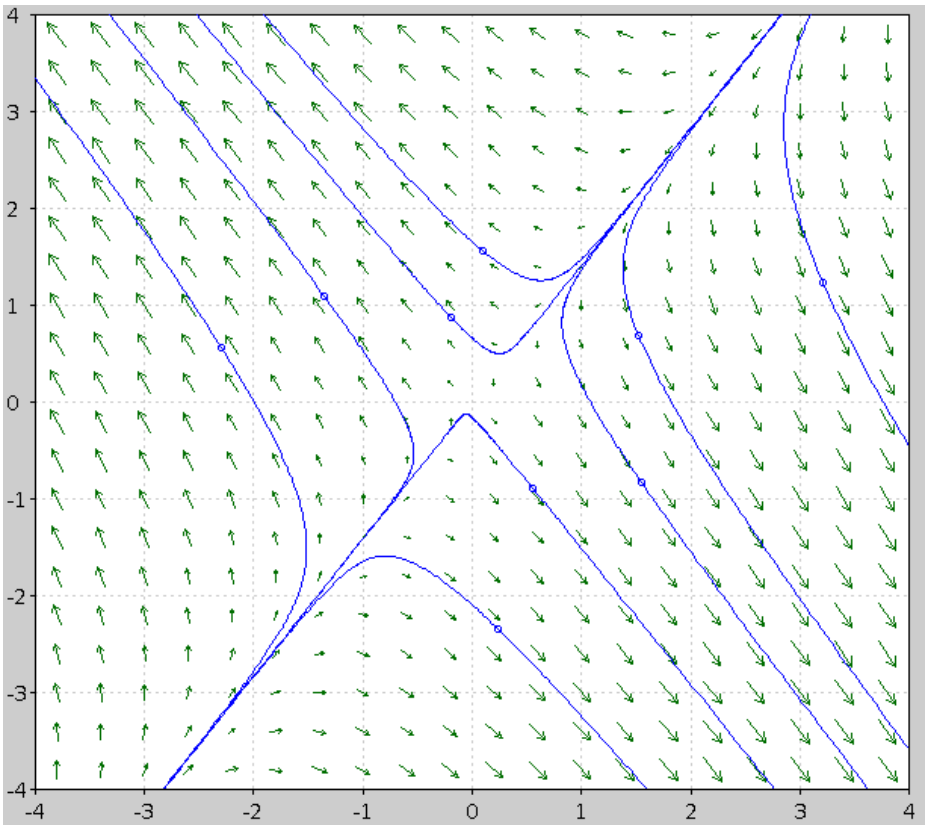
3. Which of the following systems is depicted on the phase plane?

(a) $x' = x - y,\ y' = 2x + y$

(b) $x' = x - y,\ y' = -2x + y$

(c) $x' = -x + y,\ y' = 2x - y$

(d) $x' = -x - y,\ y' = 2x + y$



4. Which of the following is a solution to the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -2 \\ -1 & -2 \end{pmatrix} \mathbf{x}?$$

(a) $\mathbf{x} = \begin{pmatrix} 2-\sqrt{6} \\ 1 \end{pmatrix} e^{-\sqrt{6}t}$

(b) $\mathbf{x} = \begin{pmatrix} \sqrt{6} \\ 1-\sqrt{6} \end{pmatrix} e^{(2-\sqrt{6})t}$

(c) $\mathbf{x} = \begin{pmatrix} \cos(\sqrt{6}t) \\ \sin(\sqrt{6}t) \end{pmatrix} e^{2t}$

(d) $\mathbf{x} = \begin{pmatrix} \sqrt{6} \\ 2+\sqrt{6} \end{pmatrix} te^{\sqrt{6}t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{\sqrt{6}t}$

5. Find the general solution to the system

$$x' = -5x - y,$$
$$y' = 4x - y.$$

6. A 3 kg mass is attached to a spring with spring constant 4 N/m and damping constant 4 N/(m/s). The spring is pulled back 1 m and then released. Find the equation describing the spring's *velocity* as a function of time.

7. Find the critical points of the system

$$x' = -2x + y^2, \quad y' = -x - 2y.$$

Say whether each critical point is asymptotically stable, stable but not asymptotically stable, or unstable.

8. You're studying some populations of snowshoe hare and lynxes (which prey on hare) that live on a nature preserve. You notice the following facts about the animals:

   (i) If the lynxes have no hare to eat, then 10% of their population dies off each month.

   (ii) If the hare are allowed to breed with no predators and unlimited resources, then their population doubles every month.

   (iii) However, the nature preserve contains limited resources that cannot support more than 2000 hare.

   (iv) When there are 1000 hare and 300 lynxes, the nature preserve is at equilibrium (meaning that neither population changes).

   Write down a system of differential equations describing the two populations.