Practice Midterm 1 MA 35100

Name:

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There are 10 points for each problem. The first two problems have to be answered in a written response and a complete argument has to be presented. For the following 8 multiple choice problems, the 10 points are achieved if the correct response is indicated on this page. Up to 5 points can be achieved if a written response on the page of the problem contains essential ideas how to solve the problem.

Please fill in your answers in the following table:

	(a)	(b)	(c)	(d)	(e)
(3)	□	□	□	□	□
(4)	□	□	□	□	□
(5)	□	□	□	□	□
(6)	□	□	□	□	□
(7)	□	□	□	□	□
(8)	□	□	□	□	□
(9)	□	□	□	□	□
(10)	□	□	□	□	□

1. Let V be a vector space. Give the definition of the dimension of V.

   Answer: The dimension of V is the number of vectors in any basis for V.

(Continue your answer on the back of the page, if needed.)

2. Let A be a 4×2 matrix. Suppose that there is a vector $B = [b_1, b_2, b_3, b_4]^T$, such that

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B$$

has a 2-dimensional solution set. Is it true that $AX = C$ always has at least one solution, for every $C = [c_1, c_2, c_3, c_4]^T$? Explain why or why not.

Approach 1: Use rank-nullity. If $AX = B$ has a 2-dimensional solution set for some specific B, then by the translation theorem, the null space of A is also 2-dimensional. Thus, $\text{nullity}(A) = 2$. By the rank-nullity theorem, $\text{rank}(A) = 0$. That is, A is the zero matrix. It is definitely not true that $0X = C$ always has at least one solution, for any C – in fact, it only has solutions if $C = 0$.

Approach 2: Since A is 4×2, its rank is at most 2. Thus, the column space of A has dimension ≤ 2. But the column space of A is the same as the set of vectors C for which $AX = C$ has at least one solution. Since the column space has dimension 2, it is strictly smaller than $\mathbb{R}^4$, which has dimension 4. So there is some C which is not in the column space. For this C, the system has no solution.

(Continue your answer on the back of the page, if needed.)

3. What is a basis of the null space of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Answer: (d). The general solution to the associated homogeneous equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(Continue your answer on the back of the page, if needed.)

4. What is true about the matrix

$$A = \begin{bmatrix} 0 & 3 & 2 \\ 2 & -7 & 4 \\ 4 & -8 & 12 \\ 5 & -4 & 19 \end{bmatrix}?$$

(a) The dimension of its column space is 1.

(b) The rank of the transpose of A equals the nullity of the transpose of A.

(c) The rank of A equals the nullity of A.

(d) The nullspace of A has dimension 2

(e) The column space is $\mathbb{R}^4$.

Answer: (b). First, calculate that the rank of A is 2 (by row-reducing, for example). Then $\text{rank}(A^T) = \text{rank}(A) = 2$. Since A^T has 4 columns, $\text{nullity}(A^T) = 4 - 2 = 2$.

Alternatively, one can eliminate the other answers. The columns of A are not all scalar multiples of a single column, so the rank of A is not 1. This rules out (a). (d) is equivalent to (a) by rank-nullity. (c) is impossible because rank + nullity = 3, an odd number. (e) is impossible because there are only three columns, which can't span $\mathbb{R}^4$.

(Continue your answer on the back of the page, if needed.)

5. What is true about the linear system

$$\begin{aligned} x + 2z &= 3 \\ y + 2z &= 3 \\ x + y + 4z &= 5 \end{aligned}$$

(a) There is no solution.

(b) There is exactly one solution and $x = 1$.

(c) There is exactly one solution and $y = 2$.

(d) There are infinitely many solutions and $x = 3 - 2z$.

(e) There are infinitely many solutions and $z = 0$.

Answer: (a). The sum of the first two equations is

$$x + y + 4z = 6,$$

which contradicts the third equation, so the system is inconsistent. One also discovers this if one does Gaussian elimination.

(Continue your answer on the back of the page, if needed.)

6. Which of the following sets of vectors span $\mathbb{R}^3$?

(a) $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$

(b) $\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}$

(c) $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(d) $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}$

(e) $\begin{bmatrix}0\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}5\\-4\\-6\end{bmatrix}$

Answer: (d). Call the vectors $D_1, \ldots, D_4$. Then

$$\frac{1}{2}(D_1 - D_2) = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

One gets the other standard basis vectors by replacing D_2 with D_3 and D_4. So all the standard basis vectors are in $\text{span}(D_1, D_2, D_3, D_4)$, so all of $\mathbb{R}^3$ is also in this span.

Alternatively: rule out (a) and (c) because one needs at least three vectors to span $\mathbb{R}^3$; rule out (b) and (e) because they are both linearly dependent, so each can only span at most a two-dimensional subspace. One has to calculate to show that (b) and (e) are linearly dependent.

(Continue your answer on the back of the page, if needed.)

7. Which of the following sets of vectors is *not* a basis of

$$\text{span}(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 14 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 14 \end{bmatrix})?$$

(a) $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 14 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 14 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$

(e) $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 14 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 14 \end{bmatrix}$

Answer: (d). In fact, neither vector in (d) is in the vector space.

Here's the most straightforward way to check this. Let the three vectors be A_1, A_2, and A_3, and let $\mathcal{V} = \text{span}(A_1, A_2, A_3)$. By solving the dependency equation for A_1, A_2, and A_3, one sees that A_3 is a linear combination of A_1

and A_2, and $\{A_1, A_2\}$ is linearly independent. Thus, A_1 and A_2 form a basis for $\mathcal{V}$, ruling out (a). So $\mathcal{V}$ is two-dimensional, which means that any set of two linearly independent vectors in $\mathcal{V}$ is a basis for $\mathcal{V}$. This rules out (c) and (e). To decide between (b) and (d), one has to calculate more. But it's not too hard to realize that, if B_1 and B_2 are the two vectors in (b), then $A_2 - A_1 = B_1$ and $A_3 - A_1 = B_2$. This rules out (b).

(Continue your answer on the back of the page, if needed.)

8. Which is a subspace of $\mathbb{R}^2$?

(a) $\{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x^2 = 0\}$

(b) $\{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $2x + 3y = 5\}$

(c) $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$

(d) $\{\begin{bmatrix} a+1 \\ b+c \end{bmatrix}$ such that $a, b, c \in \mathbb{R}\}$

(e) $\{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x^2 = y^2\}$

Answer: (a) and (d). There was a mistake and this question had two correct answers. (a) is the set $\left\{\begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R}\right\}$, which is a subspace because it's closed under addition and scalar multiplication and contains 0. Every vector in $\mathbb{R}^2$ can be written in the form of (d) for some a, b, c. So (d) $= \mathbb{R}^2$, which is a subspace of itself. Meanwhile, (b) and (c) don't contain 0, and (e) isn't closed under addition.

(Continue your answer on the back of the page, if needed.)

9. What is

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}?$$

(a) $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -3 & 0 \\ -1 & -5 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -3 \\ -1 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} -1 \\ -6 \end{bmatrix}$

(e) It is not defined.

Answer: (d). This is the only matrix with the right dimensions (2×1).

(Continue your answer on the back of the page, if needed.)

10. Which of the following is **not** a basis of the vector space of polynomials of degree at most 2?

   (a) $1, x+1, x^2+x+1$

   (b) $x+2, x^2-x+1, x^2+x+5$

   (c) $1, x, x^2$

   (d) $x+1, x+2, x^2$

   (e) x^2+x+1, x^2-2x+1, x^2-2

   Answer: (b). (b) is linearly dependent, because

$$x^2+x+5 = (x^2-x+1)+2(x+2).$$

(Continue your answer on the back of the page, if needed.)