Reduced cohomology theories:

$$\tilde{E}^* : \text{Spaces}_*^{op} \to \text{GrAb}$$

CW-complexes

① homotopy invariant

② $A \xrightarrow{i} X \longrightarrow C(i)$

$$\tilde{E}^*(A) \longleftarrow \tilde{E}^*(X) \longleftarrow \tilde{E}^*(C(i))$$

③ Sends $\vee$ to $\prod$

④ Suspension iso.

$$\tilde{E}^n(X) \xrightarrow{\simeq} \tilde{E}^{n+1}(\Sigma X)$$

For unpointed spaces, $E^*(X) = \tilde{E}^*(X \amalg *)$

$$E^*(X, A) = \tilde{E}^*(X/A)$$

$$A \xrightarrow{i} X$$

Brown representability Any such $\tilde{E}^*$ has spaces E_n such that

$$\tilde{E}^n(X) \cong [X, E_n]$$

for all pointed <u>connected</u> CW-complexes X.

<u>Proof sketch</u>.

By Yoneda, supposed to be a class

$$c_n \in \tilde{E}^n(E_n) \quad \text{such that}$$

for any X and any $d \in \tilde{E}^n(X)$,

$$\exists!\ X \xrightarrow{p} E_n \quad \text{such that} \quad p^*(c_n) = d.$$

$$E_n^{(0)} = pt.$$

Assume inductively that we have $E_n^{(r)}$ and $c_n^{(r)}$ such that for $1 \leq k \leq r$,

$$[S^k, E_n^{(r)}] \cong \tilde{E}_n(S^k)$$

① Attach $(r+1)$-spheres for each generator of $\tilde{E}_n(S^{r+1})$.

Def. A <u>spectrum</u> is a sequence of pointed spaces X_n for $n \in \mathbb{N}$, with maps

$$\Sigma X_n \longrightarrow X_{n+1}.$$

$$(X_n \longrightarrow \Omega X_{n+1}).$$

<u>ex.</u> Given any $\tilde{E}^*$, the representing spaces E_n form a spectrum.

$$E_n \xrightarrow{\sim} \Omega E_{n+1}.$$

This is called an <u>Ω-spectrum</u>.

<u>ex.</u> Suppose $K \in \mathrm{Spaces}_*$. The <u>suspension spectrum</u> of K is the spectrum $\Sigma^\infty K$ with

$$(\Sigma^\infty K)_n = \Sigma^n K.$$

$$\Sigma\Sigma^n K \xrightarrow{\simeq} \Sigma^{n+1} K.$$

$$(\Sigma^n K \longrightarrow \Omega\Sigma^{n+1} K).$$

<u>ex.</u> If $A \in \mathrm{Ab}$, the <u>Eilenberg-MacLane spectrum</u> has

$$(HA)_n = K(A, n) \qquad \left\{ \pi_* K(A,n) = \begin{array}{ll} A & \text{for } * = n \\ 0 & \text{otherwise} \end{array} \right.$$

$$K(A,n) \xrightarrow{\simeq} \Omega K(A, n+1)$$

<u>ex.</u>

$$KU_n = \begin{cases} \mathbb{Z} \times BU & n \text{ even} \\ U & n \text{ odd.} \end{cases}$$

$$U = \Omega(\mathbb{Z} \times BU).$$

$\mathbb{Z} \times BU \xrightarrow{\simeq} \Omega U$ by Bott periodicity.

<u>ex.</u>

$$KO_{8n} = \mathbb{Z} \times BO.$$

$$KO_{8n-1} = \Omega(\mathbb{Z} \times BO) \qquad \Omega^8(\mathbb{Z} \times BO) = \mathbb{Z} \times BO.$$

$$\vdots$$

$$KO_{8n-7} = \Omega^7(\mathbb{Z} \times BO)$$

$$E_n^{(r)} \vee \bigvee S^{r+1}$$

② $\tilde{E}_n(E_n^{(r)} \vee \bigvee S^{r+1}) = \hat{E}^n(E_n^{(r)}) \times \prod \hat{E}^n(S^{r+1})$.

$c_n^{(r)}$ generators

$$[S^{r+1}, E_n^{(r)} \vee \bigvee S^{r+1}] \twoheadrightarrow \tilde{E}^n(S^{r+1}).$$

③ Attach $(r+1)$-cells to kill the kernel.
This defines $E_n^{(r+1)}$.

$$E_n = \hat{\bigcup_{r=0}^{\infty}} E_n^{(r)} = \operatorname{hocolim}_{r \in \mathbb{N}} E_n^{(r)}.$$

$$\hat{E}^n(E_n) \twoheadrightarrow \lim \tilde{E}^n(E_n^{(r)}).$$

$$\begin{array}{c} X^{(r)} \\ \downarrow f_r \\ X^{(r+1)} \end{array}
\qquad
\begin{tikzcd}
\bigvee X^{(r)} \arrow[r, "id"] \arrow[d, "\bigvee f_r"] & \bigvee X^{(r)} \arrow[d] \\
\bigvee X^{(r+1)} \arrow[r] & \operatorname{hocolim} X^{(r)}
\end{tikzcd}$$

$$\bigvee X^{(r)} \longrightarrow \bigvee X^{(r)} \vee \bigvee X^{(r)} \longrightarrow \operatorname{hocolim} X^{(r)}.$$

$$\longrightarrow \underbrace{\hat{E}^*(\operatorname{hocolim} X^{(r)})} \longrightarrow \underbrace{\prod \hat{E}^*(X^{(r)}) \times \prod \hat{E}^*(X^{(r)}) \longrightarrow \prod \hat{E}^*(X^{(r)})}.$$

kernel $= \lim \hat{E}^*(X^{(r)})$.

$\boxtimes$

$$\tilde{E}^n(X) = [X, E_n]$$

$$\begin{aligned}
\tilde{E}^n(X) &\cong \tilde{E}^{n+1}(\Sigma X) \\
[X, E_n] &\cong [\Sigma X, E_{n+1}] \\
&= [X, \Omega E_{n+1}]
\end{aligned}$$

$E_n \simeq \Omega E_{n+1}$ (weak homotopy equivalence).

$[X, E_n]$ is homotopy-invariant.

$$[\bigvee X_\alpha, E_n] = \prod [X_\alpha, E_n].$$

ex. Sp = category of spectra

Sp is tensored & cotensored over $Spaces_*$.

Given $X \in Sp$, $K \in Spaces_*$,

$$(X \wedge K)_n = X_n \wedge K.$$

$$\Sigma(X_n \wedge K) = \Sigma X_n \wedge K \longrightarrow X_{n+1} \wedge K.$$

$$F(K, X)_n = F(K, X_n) \leftarrow \text{function space of spaces}$$

$$F(K, X_n) \to F(K, \Omega X_{n+1}) = \Omega F(K, X_{n+1}).$$

ex. (Thom spectra).

$BO(n)$ classifies n-dim'l real vector bundles.

ξ_n – universal bundle over $BO(n)$.

$Th(\xi_n)$ = (disk bundle of ξ_n)/(sphere bundle of ξ_n)

= ~~one-point compactification of ξ_n.~~

This is a pointed space, where the basepoint is the compactification point.

$$\begin{tikzcd} \xi_n \oplus \mathbb{R} \arrow[r] \arrow[d] & \xi_{n+1} \arrow[d] \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd}$$

$[X, BO(n)]$ $\qquad$ $[X, BO(n+1)]$

$= \mathbb{R}^n$- bundles $/X$ $\qquad$ $= \mathbb{R}^{n+1}$- bundles $/X$

$(V \to X) \longmapsto (V \oplus \mathbb{R} \to X).$

Induces $Th(\xi_n \oplus \mathbb{R}) \longrightarrow Th(\xi_{n+1}).$

$$Th(\xi_n \oplus \mathbb{R}) = \Sigma\, Th(\xi_n).$$

This defines a spectrum, MO.

Homotopy groups.

$$\pi_n X = \operatorname{colim}_r \pi_{n+r}(X_r).$$

$$\pi_{n+r}(X_r) \longrightarrow \pi_{n+r}(\Omega X_{r+1}) = \pi_{n+r+1}(X_{r+1}).$$

If E is associated to $\tilde{E}^*$,

$$\pi_{n+r}(E_r) = [S^{n+r}, E_r] = \tilde{E}^r(S^{n+r}) = \tilde{E}^0(S^n).$$

$$\pi_* \Sigma^\infty K = \pi_*^{st} K.$$

Maps of spectra.

First idea: $\operatorname{Maps}(X, Y) = \{X_n \longrightarrow Y_n\}$

making
$$\begin{array}{ccc} \Sigma X_n & \longrightarrow & X_{n+1} \\ \downarrow & & \downarrow \\ \Sigma Y_n & \longrightarrow & Y_{n+1} \end{array}$$

ex. Let $\mathbb{S} = \Sigma^\infty S^0 = \{S^0, S^1, S^2, \ldots\}$

$$\mathbb{S}^{(1)} = \{pt, S^1, S^2, \ldots\}$$

$\mathbb{S}^{(1)} \hookrightarrow \mathbb{S}$ induces an iso on π_*.

But there's no nonconstant map $\mathbb{S} \longrightarrow \mathbb{S}^{(1)}$.

ex. $\eta : S^3 \longrightarrow S^2$

$$\begin{array}{ccc} \Sigma^\infty S^3 & \longrightarrow & \Sigma^\infty S^2 \\ \| & & \| \\ \Sigma^3 \mathbb{S} & \longrightarrow & \Sigma^2 \mathbb{S} \\ \Sigma^1 \mathbb{S} & \longrightarrow & \mathbb{S} \end{array}$$

$S^1 \longrightarrow S^0$ on 0^{th} spaces.

- Restrict to <u>CW-spectra</u>.
  (all spaces CW-complexes, all maps
  $$\Sigma X_n \longrightarrow X_{n+1}$$
  CW-inclusions).
- A <u>cofinal subspectrum</u> of X is
  $$\{ A_n \hookrightarrow X_n \text{ subcomplex} \}$$
  such that
  $$\begin{tikzcd} \Sigma A_n \arrow[r, hook] \arrow[d, hook] & A_{n+1} \arrow[d, hook] \\ \Sigma X_n \arrow[r, hook] & X_{n+1} \end{tikzcd}$$
  For any cell $e_k \subseteq X_n$, some suspension $\Sigma^r e_k$ is in A_{n+r}.

A <u>map</u> $X \longrightarrow Y$ is an equivalence class of
$$\left[\begin{tikzcd} A \arrow[r, "\text{naive}"] \arrow[d, hook, "\text{cofinal subspectrum}"] & Y \\ X & \end{tikzcd} \right]$$

Possible to define homotopy using
$$X \wedge (I \amalg *) \longrightarrow Y.$$