$\mathcal{C}$ is enriched over Top_* if:

for every $X, Y \in \mathcal{C}$ there's $\mathcal{C}(X,Y) \in Top_*$

for every $X, Y, Z \in \mathcal{C}$ there's $\mathcal{C}(Y,Z) \wedge \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$ in Top_*

for every $X \in \mathcal{C}$ there's $S^0 \xrightarrow{1_X} \mathcal{C}(X,X)$

associative + unital.

- Can do this over any symmetric monoidal category.
- There's a forgetful functor

$$Top_* - Cat \longrightarrow Cat.$$

A Top_*-enrichment on $\mathcal{C}$ is a lift of $\mathcal{C}$ along this functor.

- Every $\mathcal{C}(X,Y) \ni *$ which is a zero morphism.

A tensoring of $\mathcal{C} \in Top_* - Cat$ over Top_* is

$$Top_* \times \mathcal{C} \longrightarrow \mathcal{C}$$
$$(K, X) \longmapsto K \otimes X.$$
$$Top_*(K, \mathcal{C}(X,Y)) \cong \mathcal{C}(K \otimes X, Y).$$

A cotensoring of $\mathcal{C} \in Top_* - Cat$ over Top_* is

$$Top_*^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$$
$$(K, X) \longmapsto X^K$$
$$Top_*(K, \mathcal{C}(X,Y)) \cong \mathcal{C}(X^K, Y).$$

In Sp. $Sp(X,Y) = \{\text{compatible families of maps } X_n \to Y_n\}$
$$\subseteq \prod_{n=0}^{\infty} Top_*(X_n, Y_n)$$

$$(K \wedge X)_n = K \wedge X_n$$
$$\Sigma(K \wedge X_n) = K \wedge \Sigma X_n \longrightarrow K \wedge X_{n+1}.$$
$$(X^K)_n = X_n^K$$
$$X_n^K \longrightarrow (\Omega X_{n+1})^K = \Omega(X_{n+1}^K).$$

Def. $\mathcal{M}$ is a model category which is enriched, tensored, & cotensored over Top_*.

$\mathcal{M}$ is a topological model category if:

for all $i : A \hookrightarrow B$ in Top_*

$j : X \hookrightarrow Y$ in $\mathcal{M}$,

$$i \,\square\, j : B \otimes X \coprod_{A \otimes X} A \otimes Y \longrightarrow B \otimes Y$$

is a cofibration in $\mathcal{M}$ which is acyclic if either i or j is.

Exercise: This is equivalent to:

Given $i : A \hookrightarrow B$ in Top_*,

$p : X \twoheadrightarrow Y$ in $\mathcal{M}$,

$$Hom_{\square}(i,p) : X^B \longrightarrow Y^B \times_{Y^A} X^A$$

is a fibration, which is acyclic if i or p is.

Also equivalent to:

Given $i : X \hookrightarrow Y$ in $\mathcal{M}$

$p : X' \twoheadrightarrow Y'$ in $\mathcal{M}$,

$$\mathcal{M}(Y, X') \longrightarrow \mathcal{M}(X, X') \times_{\mathcal{M}(X, Y')} \mathcal{M}(Y, Y')$$

is a fibration (in Top_*) which is acyclic if i or p is.

Other motivation:

- Model structure should be "compatible" with enrichment.
- Any model category $\mathcal{M}$ has a space of maps between objects (Dwyer-Kan or hammock localization)

These conditions imply

DK localization spaces $\simeq$ mapping spaces coming from enrichment.

$Ho(\mathcal{M})$ is enriched over $Ho(Top_*)$.

$$\mathcal{M}(Y, X') \longrightarrow \mathcal{M}(X, X') \times_{\mathcal{M}(X, Y')} \mathcal{M}(Y, Y')$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "i"', "\sim"] & X' \arrow[d, two heads, "p"] \\ Y \arrow[r] \arrow[ur, dashed] & Y' \end{tikzcd}$$

$$\begin{tikzcd} A \otimes Y \amalg_{A \otimes X} B \otimes X \arrow[r] \arrow[d, "i \,\square\, j"'] & X' \arrow[d, "q"] \\ B \otimes Y \arrow[r] \arrow[ur, dashed] & Y' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & (X')^B \arrow[d, "\mathrm{Hom}_{\square}(i, q)"] \\ Y \arrow[r] \arrow[ur, dashed] & (Y')^B \times_{(Y')^A} (X')^A \end{tikzcd}$$

Prop. Sp^{level} is a topological model category.

Pf. $A \hookrightarrow B$ in Top_*

$X \twoheadrightarrow Y$ in Sp.

$$\begin{tikzcd} F_d D^{n-1}_+ \arrow[r] \arrow[d, hook, "\sim"] & X^B \arrow[d] \\ F_d D^n_+ \arrow[r] \arrow[ur, dashed] & X^A \times_{Y^A} Y^B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} (D^{n-1}_+ \wedge B) \amalg_{D^{n-1}_+ \wedge A} (D^n_+ \wedge A) \arrow[r] \arrow[d] & X_d \arrow[d, two heads] \\ D^n_+ \wedge A \arrow[r] \arrow[ur, dashed] & Y_d \end{tikzcd}$$

this is an AC in Top_*.

■

Idea of stable model structure

Add ACs to the generators of Sp^{level}.

- Same cofs.
- More WEs (the π_*-isos).
- Fewer fibrations.

Def. $\lambda_n : F_{n+1} S^1 \longrightarrow F_n S^0$ adjoint to

$$S^1 \longrightarrow (F_n S^0)_{n+1} = S^1.$$

$$\begin{array}{cccccccccc} - & * & * & \cdots & * & * & S^1 & S^2 & S^3 & \cdots \\ & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow 1 & \downarrow & \downarrow & \\ - & * & * & \cdots & * & S^0 & S^1 & S^2 & S^3 & \cdots \end{array}$$

$$S^0 \longrightarrow *$$

$$\Sigma S^0 \longrightarrow \Sigma(*)$$

$$S^1 \longrightarrow S^1$$

λ_n is a π_*-iso, but has no homotopy inverse.

λ_n is not a cofibration.

$$\begin{tikzcd}
F_{n+1} S^1 \arrow[r, "\lambda_n"] \arrow[d, "i_1"] & F_n S^0 \arrow[d] \\
F_{n+1} S^1 \wedge [0,1]_+ \arrow[r] & Cyl(\lambda_n) \arrow[u, bend right, "\sim"'] \arrow[ul, phantom, "\ulcorner" very near start] \\
F_{n+1} S^1 \arrow[u, "i_0"] \arrow[ur, "k_n"'] &
\end{tikzcd}$$

<u>Thm.</u> There's a cofibrantly generated model structure on Sp with

$$\mathcal{W} = \pi_* \text{-isos}$$
$$I = I_{level} = \{ F_d S^{n-1}_+ \longrightarrow F_d D^n_+ \}$$
$$J = J_{level} \cup \{ k_d \,\square\, (S^{n-1}_+ \longrightarrow D^n_+) \}$$

<u>Prop.</u> $f : X \longrightarrow Y \in J^{\boxslash}$ iff

f is a levelwise fibration, and

$\forall d$, $X_d \xrightarrow{\sim} Y_d \times_{\Omega Y_{d+1}} \Omega X_{d+1}$,

<u>Cor.</u> $(Y = *)$. X is fibrant in Sp^{stable} iff.

$X_d \xrightarrow{\sim} \Omega X_{d+1}$, i.e., X is an <u>Ω-spectrum</u>.

<u>Applications.</u>

$X \longrightarrow \Omega \Sigma X$ is a π_*-iso.

$\pi_k X = \operatorname{colim} \pi_{k+n} X_n$.

$$\begin{tikzcd}
\pi_{k+n} X_n \arrow[r] \arrow[dr] & \pi_{k+n+1} \Sigma X_n \arrow[d] = \pi_{k+n} \Omega \Sigma X_n \\
& \pi_{k+n+1} X_{n+1}
\end{tikzcd}$$

$\{\pi_{k+n} \Omega \Sigma X_n\}_n$ is cofinal in $\{\pi_{k+n} X_n\}_n$.

So they have the same colimit.

- $\pi_n \Sigma X = \pi_{n-1} X$

  $\pi_n \Omega X = \pi_{n+1} X$. $\implies \Sigma, \Omega$ preserve π_*-isos.

- Σ preserves cofibrations.

$X \xrightarrow{\sim} \Omega\Sigma X$ $\qquad \Sigma : Sp^{stable} \rightleftarrows Sp^{stable} : \Omega$

This is a Quillen equivalence.

Σ, Ω are inverse equivalences on $Ho(Sp)$.

- $\Sigma^\infty = F_0 : Top_* \longrightarrow Sp$

preserves cofs, & sends WEs to levelwise WEs.

(which are stable WEs, too.)

$$\Sigma^\infty : Top_* \rightleftarrows Sp : Ev_0.$$

The homotopy-invariant concept is

$$\mathbb{R}(Ev_0)(X) := Ev_0(X^{fib})$$
$$= \Omega^\infty X.$$

This is an infinite loop space.

$(X^{fib})_n = \mathrm{hocolim}\ \Omega^k X_{n+k}.$

$$\begin{tikzcd}
F_{n+1}S^1 \arrow[r, "\lambda_n"] \arrow[d, "i_1"] & F_n S^0 \arrow[d] \\
F_{n+1}S^1 \wedge [0,1]_+ \arrow[r] & Cyl(\lambda_n) \arrow[u, bend right, "\sim"'] \\
F_{n+1}S^1 \arrow[u, "i_0"] \arrow[ur, "\underline{k_n}"] &
\end{tikzcd}$$

n^{th} space:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^0 \arrow[d] \\
* \arrow[r] & S^0 \\
* \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$n+k^{th}$ space:

$$\begin{tikzcd}
S^k \arrow[r, "1"] \arrow[d] & S^k \arrow[d] \\
\frac{S^k \times [0,1]}{* \times [0,1]} = S^k \wedge [0,1]_+ \arrow[r] & S^k \wedge [0,1]_+ \\
S^k \arrow[u] \arrow[ur, "i_0."] &
\end{tikzcd}$$

$$\begin{tikzcd}
k_n & * \arrow[d] & \cdots & * \arrow[d] & * \arrow[d] & S^1 \arrow[d] & S^2 \arrow[d] & \cdots & A \arrow[d] \\
& * & & * & S^0 & S^1 \wedge [0,1]_+ & S^2 \wedge [0,1]_+ & & B
\end{tikzcd}$$

$$k_n \square (S^{n-1}_+ \longrightarrow D^n_+) = A \wedge D^n_+ \coprod_{A \wedge S^{n-1}_+} B \wedge S^{n-1}_+ \longrightarrow B \wedge D^n_+.$$

n^{th} stage: $S^{n-1}_+ \longrightarrow D^n_+$

$(n+k)^{th}$ stage: $S^k \wedge D^n_+ \coprod_{S^k \wedge S^{n-1}_+} S^k \wedge [0,1]_+ \wedge S^{n-1}_+ \longrightarrow S^k \wedge [0,1]_+ \wedge S^{n-1}_+.$