Outline for a course in stable homotopy theory

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This will be a semester-long course, held over Zoom, time TBA. I've tried to break it down into discrete talks that should take an hour each. Get in touch if you'd like to give one of the talks; I'll give the rest. If you're a grad student, you should strongly consider at least one talk – that's at least one thing from the class that you're guaranteed to actually learn.

Resources. We will be following [BR] for the bulk of the course. Among other things, this book surveys most of the structured categories of spectra now in use.

I first learned this from part III of Adams's "blue book" [Ada]. (Note that parts I and II are more advanced and should be skipped at first.) At the time this was written, Adams did not have access to any structured category of spectra, and you can see the consequences in his discussion of the smash product in chapter 4. But the rest is maybe a testament to what you can do with just the stable homotopy category.

For stable ∞-categories, see chapter 1 of [Lur]. For some applications of monoidal ∞-category technology to spectra, see chapter 7 of [Lur].

For the various structured categories of spectra, see [EKMM], [LMS], [HSS], [MMSS], and [Sch]. I think [Sch] is the only one of these appropriate to someone learning this for the first time (and a few of you have already spent some time with it).

1. **Introduction.** What are spectra meant to do? They generalize four ideas simultaneously. First, they contain the category of "spaces up to arbitrarily high suspension", so allow the isolation of properties of spaces that are "stable" under suspension. Second, the Brown representability theorem states that every cohomology theory is represented by a spectrum. Third, a spectrum can be thought of as an infinite loop space with a choice of infinitely many deloopings, and there are many geometrically interesting infinite loop spaces. Finally, the category of spectra Sp is something like the category of chain complexes over a ring $\mathsf{Ch}(R)$, and is really a universal example of a "category like $\mathsf{Ch}(R)$". Besides the applications to geometric topology this machinery was originally developed for, in recent years there has been heavy interest in doing algebraic geometry with highly structured ring spectra, and in working with other "stable homotopy categories" like Sp. (I will give this talk.)

2. **Review of unstable homotopy theory.** Generalized homology and cohomology theories; cofiber and fiber sequences of spaces, and the long exact sequences they induce on homology or homotopy; the Freudenthal suspension theorem; the definition of stable homotopy groups; and whatever else people want to review. Reference: [BR, 1.1], [Ada, III.1].

3. **The homotopy category of sequential spectra, and some examples.** Suspension spectra of spaces, Ω-spectra, Eilenberg-MacLane spectra, and K-theory. Reference: [BR, 1.2], [Ada, III.2].

4. **Review of model categories.** Constructions: the homotopy category, Quillen adjunctions and derived functors, Quillen equivalences. Some examples: model categories of spaces, simplicial sets, and chain complexes over a ring. Reference: [DS].

5. **The levelwise model structure on sequential spectra.** Reference: [BR, 2.1].

6. **Homotopy groups of spectra.** Define them and give some examples. Reference: [BR, 2.2].

7. **The stable model structure on sequential spectra.** This is defined by inverting π_*-isomorphisms, i.e., forcing homotopy groups to detect weak equivalences. Check that it recovers the homotopy category we originally defined. Reference: [BR, 2.3].

8. **Suspensions and loops of spectra.** Define them and show that they are inverse to each other on the homotopy category. Since any ΣX is a cogroup object and ΩX is a group object in $\mathrm{Ho}(\mathsf{Sp})$, for any X, it follows that any set of homotopy classes of maps $[X, Y]_*$ is in fact an abelian group. Reference: [BR, 3.1,3.2].

9. **Fiber and cofiber sequences of spectra.** The big point here is that fiber sequences *are* cofiber sequences, and can be extended both backward and forward. Thus, one can define long exact sequences of groups of homotopy classes of maps in multiple equivalent ways. Reference: [BR, 3.3-3.6], though you do not need to go into their level of detail. (In particular, the fact that fiber sequences can be extended backwards, and cofiber sequences forwards, in any model category is a generalization of a fact we've already discussed about spaces, and I don't think we need to give the fully general argument.)

10. **Other properties of spectra.** Products and wedge sums, connective covers and truncations, the Σ^∞-Ω^∞ adjunction. and Brown representability. Optionally, talk about the Milnor exact sequence describing π_* of an inverse limit in [Ada, III.8]. Reference: [Ada, III.3].

11. **The stable homotopy category as a triangulated category.** Define triangulated categories, and show how this definition is a natural consequence of the properties we've examined. Likewise talk about the other major example: homotopy categories of chain complexes over a ring. Reference: [BR, 4.1-4.2], specialized to these cases.

12. **The Atiyah-Hirzebruch spectral sequence.** This should start with a review of spectral sequences in general. As an application, compute $K^*(\mathbb{C}P^n)$. Reference: [Ada, III.7]; for spectral sequences see [MT, ch. 7] or [McC].

13. **The smash product.** Construct the smash product on sequential spectra, as in [BR, 6.2], [Ada, III.4]. This only has good properties (only defines a symmetric monoidal structure) on the stable homotopy category. In as much detail as you'd like, discuss the concept of a monoidal model category, and the existence of solutions ot this problem

such as the model category of symmetric spectra. References: [BR, 5.2, 6.1-6.2], [Ada, III.4].

14. **Structured model categories of spectra, *or* the stable ∞-category of spectra.** We can cover one or both of these depending on your interest. Sequential spectra are bad if you care about anything like a symmetric monoidal structure, or spectra with group actions. Various solutions to these problems were constructed from the 80s to the 2000s, and it's at least fairly simple to talk about symmetric spectra and orthogonal spectra. The stable ∞-category perspective is even more modern, and has the advantage of solving all problems at once as long as the problems are framed ∞-categorically. Unfortunately, this framing can take a fair bit of work. (I'll give this talk, unless someone else is really gung-ho about it.) References: [BR, 5.2-5.3], [Lur, ch. 1].

15. **Applications of the smash product.** Spanier-Whitehead duality, homology, and the internal function spectrum. Reference: [BR, 6.3, 6.4.2, 6.5].

16. **Ring spectra.** Define these at the level of the homotopy category, and note that the homotopy groups of a ring spectrum are a graded ring. The homotopy groups of a homotopy commutative ring spectrum are graded-commutative, i.e., odd-degree elements anti-commute: why? Some example: the sphere, the Eilenberg-MacLane spectrum HR for a ring R, and KO and KU. If E is a ring spectrum, then $E^*(X)$ is a ring for any *space* X. This uses the fact that X has a diagonal map, which translates to a *coalgebra* structure on its suspension spectrum. Reference: [BR, 6.6].

17. **Thom spaces, Thom spectra, and cobordism theories.** Define the Thom spectrum of a vector bundle, and use this to construct the cobordism theory MO as a spectrum. Discuss cobordism theories with extra structure, notably complex and framed cobordism. Show that framed cobordism is equivalent to the sphere spectrum (the Pontryagin-Thom construction). Reference: [May2, ch. 25].

18. **The Steenrod algebra and its dual.** Without necessarily proving everything, let's talk about the properties of the Steenrod algebra, and its spectral definition $\mathcal{A} = H\mathbb{F}_p^*H\mathbb{F}_p$. Using spectra, prove that $\mathcal{A}$ is a cocommutative (but not commutative) Hopf algebra, and that for any X, $H\mathbb{F}_p^*X$ is a module over this Hopf algebra. Milnor realized that it was easier to work with the *dual* Steenrod algebra $\mathcal{A}_* = (H\mathbb{F}_p)_*H\mathbb{F}_p$. This is a commutative (but not cocommutative) Hopf algebra, which Milnor showed is free as a graded-commutative $\mathbb{F}_p$-algebra, and for any X, $(H\mathbb{F}_p)_*X$ is a comodule over it. Reference: [BR, 2.5].

19. **The Adams spectral sequence.** Construct it, interpret its E_2 page as Ext of $\mathcal{A}$-modules and $\mathcal{A}_*$-comodules, and show that it converges to the p-complete stable homotopy groups of any finite type CW-complex X. This would be a good time to review any necessary homological algebra, e.g. what Ext is. Reference: [BR, 2.6].

20. **Basic computations with the Adams spectral sequence.** We should be able to compute the homotopy groups of MO, bu, and bo using the spectral sequence and change-of-rings theorems. Reference: [Rav, 3.1].

21. **Computing homotopy groups of spheres with the Adams spectral sequence.** Let's focus on the prime 2 and the range up to about the 14 stem (the first nontrivial differential). The hardest part is to get the E_2 term, and there are a few ways to do it: the May spectral sequence obtained by filtering the dual Steenrod algebra [Rav, 3.2], directly constructing a minimal $\mathcal{A}$-module resolution in the desired range [Bru], or using a computer program. Pick your favorite and show us. Talk about some of the multiplicative structure, and the Hopf invariant 1 elements (those appearing on the 1-line).

22. **The generalized Adams spectral sequence.** Define the E-nilpotent completion of a spectrum, $X_E^\wedge$ and construct the E-based Adams spectral sequence computing $\pi_* X_E^\wedge$. Reference: [Rav, 3.3].

23. **Operads.** Define operads for spaces and show how they can also be made to act on spectra in a monoidal model category. Focus on the examples of E_n, A_∞ (also known as E_1), and E_∞. Reference: [May1], though you may want to look for newer ones.

24. E_∞ **ring spectra.** Lots of examples (every ring spectrum we've looked at so far!). One easy way to construct them: as strict commutative monoids in symmetric spectra. Also note that the suspension spectrum of a space is an E_∞-coalgebra, so that for any ring spectrum R, $F(\Sigma^\infty_+ X, R)$ is an E_∞-algebra.

25. **Power operations.** The main point is that any E_∞-ring spectrum R has external power operations $R^*(X) \to R^*(X^{\times d}_{\Sigma_d})$, and internal ones $R^*(X) \to R^*(X)$ given by composing with the diagonal map of X. Let's talk about as many examples we can: Steenrod operations on ordinary cohomology, Adams operations on K-theory, and power operations on cobordism following Quillen [Qui].

References

[Ada] Adams, J. F. *Stable Homotopy and Generalised Homology.* Chicago Lectures in Mathematics Series (1974).

[BR] Barnes, D. and C. Rotzheim. *Foundations of Stable Homotopy Theory.* Cambridge Studies in Advanced Mathematics **185** (2020).

[Bru] Bruner, R. B. "An Adams Spectral Sequence Primer". Available online at `http://www.rrb.wayne.edu/papers/adams.pdf` (2009).

[DS] Dwyer, Dwyer, W. G., and J. Spalinski. "Homotopy theories and model categories." In *Handbook of algebraic topology* (1995): 73–126.

[EKMM] Elmendorf, A. D., I. Kriz, M. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory.* American Mathematical Society Surveys and Monographs (1995).

[HSS] Hovey, M., B. Shipley, and J. Smith. "Symmetric spectra." *Journal of the American Mathematical Society* **13**.1 (2000): 149–208.

[LMS] Lewis, G., J. P. May, and M. Steinberger. *Equivariant stable homotopy theory.* Lecture Notes in Mathematics **1213** (1986).

[Lur] Lurie, J. *Higher Algebra*. Unpublished, available online at `http://people.math.harvard.edu/~lurie/papers/HA.pdf`.

[MMSS] Mandell, M. A., J. P. May, B. Shipley, and J. Smith. "Model categories of diagram spectra." *Proceedings of the London Mathematical Society* **82**.2 (2001): 441–512.

[May1] May, J. P. *The Geometry of Iterated Loop Spaces.* Lecture Notes in Mathematics **271** (1972).

[May2] May, J. P. *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics Series (1999).

[McC] McCleary, J. *A User's Guide to Spectral Sequences.* Cambridge Studies in Advanced Mathematics **58** (1985).

[MT] Mosher, R. E. and M. C. Tangora. *Cohomology Operations and Applications in Homotopy Theory.*

[Qui] Quillen, D. "Elementary proofs of some results in cobordism theory using Steenrod operations." *Advances in Mathematics* **7**: 29–56 (1971).

[Rav] Ravenel, D. C. *Complex Cobordism and Stable Homotopy Groups of Spheres.* (2003).

[Sch] Schwede, S. *An untitled book project about symmetric spectra*. Unpublished, available online at `http://proxy.math.uni-bonn.de/people/schwede/SymSpec.pdf`.