Lecture 1: Introduction and the Steenrod algebra

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This class concerns the Sullivan conjecture, and more generally following basic setup: given a finite CW-complex X and a finite p-group π acting on X, it's possible to describe the fixed point set X^π in terms of homotopy theory. This is surprising because X^π isn't even homotopy invariant! For instance, if $\mathbb{Z}$ is acting trivially on a point, the fixed point set is the whole point... but replacing the point with the homotopy-equivalent space $\mathbb{R}$, with $\mathbb{Z}$ acting by translation $x \mapsto x+1$, the fixed point set is empty.

The basic tools we'll use are modules and algebras over the Steenrod algebra $\mathcal{A}$, and in particular, homological algebra of unstable modules. A first goal is a theorem of Carlsson that says that $\widetilde{H}^*(\mathbb{R}P^\infty)$ is *injective* as an $\mathcal{A}$-module. Why should this be true? We'll also use some basic *unstable* homotopy theory, in particular the Bousfield-Kan "unstable Adams spectral sequence."

Example 1. When X is a sphere S^n, the basic setup is called 'Smith theory,' after P. A. Smith, one of a number of homotopy-theoretic Smiths. Let $\pi = \mathbb{Z}/2 = \{1, \tau\}$. Writing

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\},$$

we can have $\mathbb{Z}/2$ act by

$$\tau(x_0, \dots, x_n) = (x_0, \dots, x_i, -x_{i+1}, \dots, -x_n).$$

The fixed points of this action are $\{(x_0, \dots, x_i, 0, \dots, 0) : x_0^2 + \cdots + x_i^2 = 1\} = S^i$. So we can get any lower-dimensional sphere as a fixed point space of such an action. Smith theory says that, up to homology, this is all we see.

Theorem 2. *If X is a finite CW-complex with $H_*(X; \mathbb{Z}/p) \cong H_*(S^{2n+1}; \mathbb{Z}/p)$, and the cyclic group C_p acts on X, then $H_*(X^{C_p}; \mathbb{Z}/p) \cong H_*(S^i; \mathbb{Z}/p)$ for some i, where i is odd if p is odd.*

Note that the input and output of this theorem are both in terms of homology. This will turn out to happen a lot.

Cohomology and the Steenrod algebra

If X is a space and k is a field, then by the Künneth formula, $H^*(X; k)$ is a graded commutative ring. **Graded commutative** doesn't mean commutative – it means that for homogeneous elements x and y of degrees $|x|$ and $|y|$,

$$xy = (-1)^{|x||y|} yx.$$

In particular, if $|x|$ is odd, $x^2 = -x^2$, so in characteristic not equal to two, everything squares to 0.

In characteristic 0, this ring structure is all the natural structure on cohomology. But in characteristic p, there's more!

Say $k = \mathbb{F}_2$. The **Steenrod squares** are natural abelian group homomorphisms

$$\mathrm{Sq}^i : H^n(X; \mathbb{F}_2) \to H^{n+i}(X; \mathbb{F}_2).$$

They satisfy a few axioms.

1. $\mathrm{Sq}^0(x) = x$. (Note for experts: for fields larger than $\mathbb{F}_2$, this map should actually be the Frobenius.)

2. $\mathrm{Sq}^n(x) = x^2$ if $|x| = n$, and $\mathrm{Sq}^i(x) = 0$ if $i > |x|$ (the unstable condition).

3. $\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y)$ (the Cartan formula).

4. $\mathrm{Sq}^i \mathrm{Sq}^j = \sum_t \binom{j-t-1}{i-2t} \mathrm{Sq}^{i+j-t} \mathrm{Sq}^t$ (the Adem relations).

The binomial coefficient appearing in the Adem relations is a mod 2 binomial coefficient, and can be defined even for $n \le 0$ by the power series identity

$$(1+x)^n = \sum_{i\ge 0} \binom{n}{i} x^i.$$

In particular, $\binom{n}{i} = 0$ for $i < 0$, so the sum in the Adem relations only uses $0 \le t \le \lfloor \frac{i}{2} \rfloor$.

We'll construct these later. They're actually totally specified by the above axioms (and naturality), so we can play around with them right now.

Example 3. For $X = \mathbb{R}P^\infty$, then $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$ with $|x| = 1$. We must have $\mathrm{Sq}^0(x) = x$ and $\mathrm{Sq}^1(x) = x^2$, and the higher squares must vanish by the unstable condition.

This actually tells us how the squares act on powers of x, too, but to prove this, it's worth rewriting the Cartan formula. Define $\mathrm{Sq}(y) = \sum \mathrm{Sq}^i(y)$ – then the Cartan formula says that Sq is a ring homomorphism.

Returning to the case of $\mathbb{R}P^\infty$, we have $\mathrm{Sq}(x) = x + x^2$, so

$$\mathrm{Sq}(x^n) = (\mathrm{Sq}\, x)^n (x+x^2)^n = x^n(1+x)^n = \sum \binom{n}{i} x^{n+i}.$$

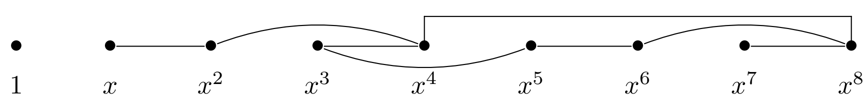
Thus,

$$\mathrm{Sq}^i(x^n) = \binom{n}{i} x^{n+i}.$$

In particular,

$$\mathrm{Sq}^1(x^n) = nx^{n+1} = \begin{cases} 0 & n \text{ even} \\ x^{n+1} & n \text{ odd.} \end{cases}$$

We can start to draw this in a diagram like the following. Here, each dot represents a cohomology class, the straight lines are Sq^1s, the curved lines Sq^2s, and the long, square-shaped line a Sq^4.



Why haven't we drawn Sq^3? Because of the Adem relations:

$$\mathrm{Sq}^3 = \sum \binom{1-t}{1-2t} \mathrm{Sq}^{3-t} \mathrm{Sq}^t = \mathrm{Sq}^3.$$

Here's another one, for practice:

$$\mathrm{Sq}^2 \mathrm{Sq}^4 = \sum \binom{3-t}{2-2t} \mathrm{Sq}^{6-t} \mathrm{Sq}^t = \mathrm{Sq}^6 + \mathrm{Sq}^5 \mathrm{Sq}^1 = \mathrm{Sq}^6 + \mathrm{Sq}^1 \mathrm{Sq}^4 \mathrm{Sq}^1.$$

Thus, $\mathrm{Sq}^6 = \mathrm{Sq}^2 \mathrm{Sq}^4 + \mathrm{Sq}^1 \mathrm{Sq}^4 \mathrm{Sq}^1$. This illustrates an important point: we can write any square as a sum of products of Sq^{2^i} for $i \ge 0$.

The Steenrod squares are **stable** under the unreduced suspension $X \mapsto \Sigma X$. That is, for the natural isomorphism

$$\sigma : \widetilde{H}^n(X) \xrightarrow{\sim} \widetilde{H}^{n+1}(\Sigma X),$$

we have $\sigma\, \mathrm{Sq}^n(x) = \mathrm{Sq}^n(\sigma x)$.

Example 4. Let $h : S^3 \to S^2$ be the Hopf map. The cone on h is $S^2 \cup_h e^4 \simeq \mathbb{C}P^2$. This has cohomology $H^*\mathbb{C}P^2 = \mathbb{F}_2[y]/(y^3)$, with $|y| = 2$. Thus, $\operatorname{Sq}^2 y = y^2 \neq 0$. Now, if $X = \Sigma^n \mathbb{C}P^2$, we no longer have that $\sigma^n(y)^2 = \sigma^n(y^2)$, for obvious degree reasons. But we do have $\operatorname{Sq}^2(\sigma^n y) = \sigma^n(\operatorname{Sq}^2 y) = \sigma^n(y^2) \neq 0$. This is a nontrivial square in

$$\Sigma^n(S^2 \cup_h e^4) = S^{n+2} \cup_{\Sigma^n h} e^{n+4}.$$

This means that this cone can't split up as a wedge, which means that $\Sigma^n h$ is never nullhomotopic. We've used the Steenrod squares to find a stable homotopy element.

Proof of stability of the squares. There's a quotient map $S^1 \times X \to \Sigma X$ inducing a map

$$H^*\Sigma X \to H^*S^1 \otimes H^*X = H^*X \oplus e_1 H^{*-1}X,$$

and $H^*\Sigma X$ is mapped isomorphically to the summand $e_1 H^{*-1}X$. We can use the Cartan formula to calculate squares here – since $\operatorname{Sq}^i e_1 = 0$ for $i > 0$, they correspond precisely to the squares in H^*X. $\square$