Lecture 10: The T-functor of algebras

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We're talking about the Lannes T-functor, $T_V : \mathcal{U} \to \mathcal{U}$, with

$$\mathrm{Hom}_{\mathcal{U}}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^*BV \otimes N)$$

Example 1. Let $M = F(n)$, the free unstable module on a single generator $i_n \in F(n)^n$. For simplicity, take $p = 2$ and $V = \mathbb{F}_2$, so $H * BV = H^*\mathbb{R}P^\infty \cong \mathbb{F}_2[u]$ with u in degree 2. The adjunction formula above is

$$\mathrm{Hom}_{\mathcal{U}}(T_V F(n), N) \cong \mathrm{Hom}_{\mathcal{U}}(F(n), \mathbb{F}_2[u] \otimes N) \cong (\mathbb{F}_2[u] \otimes N)^n = \bigoplus_{i=0}^{n} N^{n-i}.$$

By the Yoneda lemma, we get $T_V F(n) = \bigoplus_{i=0}^n F(n-i)$.

Let's suppose we know that $T_V U(M) \cong U T_V(M)$, where $U : \mathcal{U} \to \mathcal{K}$ is the left adjoint to the forgetful functor (in fact,

$$U(M) = \mathrm{Sym}(M)/(\mathrm{Sq}^{|x|}(x) = x^2).)$$

By a theorem of Serre, $H^*K(\mathbb{Z}/2, n) \cong U(F(n))$. Thus,

$$TH^*K(\mathbb{Z}/2, n) = U\left(\bigoplus_{i=0}^{n} F(n-i)\right) = \bigotimes_{i=0}^{n} H^*K(\mathbb{F}_2, n-i).$$

What's $\mathrm{map}(\mathbb{R}P^\infty, K(\mathbb{Z}/2, n))$? Since this is a commutative topological group, it's a product of Eilenberg-Mac Lane spaces. We have

$$\pi_i \,\mathrm{map}(\mathbb{R}P^\infty, K(\mathbb{Z}/2, n)) = [\mathbb{R}P^\infty, K(\mathbb{Z}/2, n-i)] = H^{n-i}(\mathbb{R}P^\infty, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \le i \le n \\ 0 & i > n. \end{cases}$$

Thus,

$$\mathrm{map}(\mathbb{R}P^\infty, K(\mathbb{Z}/2, n)) = \prod_{i=0}^{n} K(\mathbb{Z}/2, n-i),$$

with cohomology

$$H^* \mathrm{map}(\mathbb{R}P^\infty, K(\mathbb{Z}/2, n)) \cong TH^*K(\mathbb{Z}/2, n).$$

Let's get back to technical stuff, including establishing the claim that T_V commutes with U. Given $M \in \mathcal{U}$, $1 : T_V M \to T_V M$ has an adjoint map

$$\epsilon_M : M \to H^*BV \otimes T_V M.$$

There's then a map

$$M \otimes N \overset{\epsilon_M \otimes \epsilon_M}{\to} H^*BV \otimes T_V M \otimes H^*BV \otimes T_V N \overset{\text{swap}}{\to} H^*BV \otimes H^*BV \otimes T_V M \otimes T_V N \overset{m \otimes 1}{\to} H^*BV \otimes T_V M \otimes T_V N.$$

This is adjoint to a map

$$T_V(M \otimes N) \to T_V M \otimes T_V N.$$

Theorem 2. *This map is an isomorphism.*

Proof sketch. Let $T = T_{\mathbb{F}_p}$; since T_V is a composition of copies of T, we can assume $T_V = T$. Since T is exact and preserves sums, we can also assume $M = F(p)$ and $N = F(q)$. The proof is then done by an elaborate multiple induction on p, q, and the internal degree of an element of the tensor product. This goes by looking at the exact sequences

$$0 \to F(p) \otimes \Phi F(q) \stackrel{1 \otimes \lambda}{\to} F(p) \otimes F(q) \to F(p) \otimes \Sigma\Omega F(q) \to 0$$

and

$$0 \to \Phi F(p) \otimes \Phi F(q) \stackrel{\lambda \otimes 1}{\to} F(p) \otimes \Phi F(q) \to \Sigma\Omega F(p) \otimes \Phi F(q) \to 0.$$

We note that $\Omega F(p) = F(p-1)$, allowing for the induction, and that T is exact and commutes with Σ and Φ, so we can apply it to the above exact sequences to start the proof. $\square$

Remark 3. That T commutes with Φ implies that

$$TH^*\mathbb{C}P^\infty = T\Phi H^*\mathbb{R}P^\infty = \Phi T H^*\mathbb{R}P^\infty = H^*\mathbb{C}P^\infty \otimes H^*K(\mathbb{Z}/2, 0).$$

Since T_V commutes with tensor products, $T_V K \in \mathcal{K}$ if $K \in \mathcal{K}$.

Theorem 4. *T_V is also left adjoint to $H^*BV \otimes \cdot$ in $\mathcal{K}$; that is,*

$$\mathrm{Hom}_{\mathcal{K}}(T_V K, L) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*BV \otimes L).$$

(Dylan: a high-falutin reason for this is that T_V commutes with sifted colimits, which agree in $\mathcal{K}$ and $\mathcal{U}$ by monadicity, and with tensor products, which are coproducts in $\mathcal{K}$. So it commutes with all colimits in $\mathcal{K}$ and we can apply the adjoint functor theorem.

PG: the below is an unpacking of this reason.)

Definition 5. A reflexive coequalizer diagram is a diagram

$$C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0, \qquad C_1 \xleftarrow{s_0} C_0$$

where $d_0 s_0 = d_1 s_0 = 1$.

Proposition 6. *Let*

$$K_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} K_0, \qquad K_1 \xleftarrow{s_0} K_0$$

be a reflexive coequalizer diagram in $\mathcal{K}$. Then the coequalizer in $\mathcal{K}$ is isomorphic to the coequalizer in $\mathcal{U}$.

Proof. The coequalizer in $\mathcal{U}$ is $K_0/\partial K_1$ where $\partial = d_1 - d_0$. If $a \in K_0$, then

$$a\partial(y) = a(d_1 y - d_0 y) = d_1(s_0(a)y) - d_0(s_0(a)y) \in \partial K_1,$$

so ∂K_1 is an ideal of K_0, and thus the coequalizer is canonically an algebra. $\square$

Lemma 7. *If $K \in \mathcal{K}$, there's a reflexive coequalizer diagram*

$$U(F_1) \underset{d_1}{\overset{d_0}{\rightrightarrows}} U(F_0) \longrightarrow K, \qquad U(F_1) \xleftarrow{s_0} U(F_0)$$

where F_0 and F_1 are projective in $\mathcal{U}$.

Proof. Recall the chain of adjunctions

$$\mathcal{K} \overset{U}{\rightleftarrows} \mathcal{U} \overset{F}{\rightleftarrows} \mathsf{GradedVectorSpaces}.$$

Let $G = UF$. Then if $K \in \mathcal{K}$, we have maps $\epsilon : K \to G(K)$ of vector spaces and $d_0 : G(K) \to K$ of algebras such that $d_0\epsilon = K$. We have a coequalizer diagram

$$G^2K \underset{d_0G=d_1}{\overset{Gd_0}{\rightrightarrows}} G(K) \xrightarrow[d_0]{} K,$$

and ϵ induces the inverse of $d_0 : G(K)/\partial G^2(K) \to K$. Thus, $G\epsilon = s_0$ completes the reflexive coequalizer diagram, and we can take $F_0 = F(K)$, $F_1 = FG(K)$. (In fact, the coequalizer diagram extends to a simplicial object in an obvious way, and ϵ is a simplicial contraction of this simplicial object.) □

Lemma 8. *The functor $\mathcal{K} \to \mathcal{K}$ given by $L \mapsto H^*BV \otimes L$ has a left adjoint $\widetilde{T}_V$ on $\mathcal{K}$.*

Proof. Define $\widetilde{T}_V(U(M)) = UT_V(M)$. Since every K is a reflexive coequalizer of objects $U(M)$, and since reflexive coequalizers are preserved by all functors, this defines $\widetilde{T}_V$ for all K. TO be precise,

$$\widetilde{T}_V(K) = \pi_0\widetilde{T}_V(G_0K) := T_VGK/\partial TG^2K.$$

We have

$$\mathrm{Hom}_{\mathcal{K}}(\widetilde{T}_V(U(M)), L) \cong \mathrm{Hom}_{\mathcal{K}}(U(T_VM), L) \cong \mathrm{Hom}_{\mathcal{U}}(T_VM, L) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^*BV \otimes L) \cong \mathrm{Hom}_{\mathcal{K}}(U(M), H^*BV \otimes L).$$

Thus, this follows for all K, for the same reason that reflexive coequalizers are absolute. □

To prove that $T_V \cong \widetilde{T}_V$, we now just have to prove:

Lemma 9.

$$\widetilde{T}_VU(M) = UT_V(M) \cong T_VU(M).$$

Proof. There are maps in $\mathcal{U}$, $M \to U(M)$ and $T_VM \to U(T_VM)$. The first of these gives a map $T_VM \to T_VU(M)$ in $\mathcal{U}$, and thus $U(T_VM) \to UT_VU(M) \to T_VU(M)$ in $\mathcal{K}$. The second gives

$$M \to H^*BV \otimes T_VM \to H^*BV \otimes U(T_VM)$$

in $\mathcal{U}$, and thus

$$U(M) \to H^*BV \otimes U(T_VM)$$

in $\mathcal{U}$ (forgetting again). Taking the adjunction in $\mathcal{U}$ gives

$$T_VU(M) \to U(T_VM)$$

and then one shows that these two maps are inverse in $\mathcal{U}$, and thus in $\mathcal{K}$. □