Lecture 11: Cohomology of mapping spaces

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Recall that we had this functor $T_V : \mathcal{K} \to \mathcal{K}$ satisfying

$$\mathrm{Hom}_{\mathcal{K}}(T_V K, L) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*BV \otimes L).$$

Example 1. If X and Y are good spaces, then

$$\mathrm{Hom}_{\mathcal{K}}(T_V H^*X, H^*Y) \cong \mathrm{Hom}_{\mathcal{K}}(H^*X, H^*BV \otimes H^*Y).$$

Let $Y = \mathrm{map}(BV, X)$. Then evaluation

$$\begin{aligned} \mathrm{map}(BV, X) \times BV &\longrightarrow X \\ (f, x) &\mapsto f(x) \end{aligned}$$

induces

$$H^*X \to H^*\mathrm{map}(BV, X) \otimes H^*BV$$

and thus, by adjunction,

$$T_V H^*X \to H^*\mathrm{map}(BV, X)$$

in $\mathcal{K}$. This map is often an isomorphism, and we're about to start exploring when it is. If you didn't already know, computing cohomology of mapping spaces is a tough business, so it's nice that we're able to do this.

Extended example: Maps between classifying spaces

This area was deeply explored by Wilkerson. Let G be a group and BG the classifying space, so that $\Omega BG \simeq G$. If G is discrete, this must satisfy $\pi_1 BG \cong G, \pi_n BG = 0$ for $n \geq 2$. By covering space theory, if X is a connected pointed CW-complex and G is discrete, then $[X, BG]_* \cong \mathrm{Hom}_{\mathsf{Gp}}(\pi_1 X, G)$. If we instead look at unpointed maps, we get

$$[X, BG] \cong \mathrm{Hom}_{\mathsf{Gp}}(\pi_1 X, G)/\text{conjugacy in } G =: \mathrm{Rep}(\pi_1 X, G).$$

In particular, we can prove

Theorem 2. *If H and G are discrete, then the space of unpointed maps*

$$\mathrm{map}(BH, BG) = \coprod_{\rho \in \mathrm{Rep}(H,G)} BC(\rho),$$

where the ***centralizer*** *$BC(\rho) \subseteq G$ is the subgroup of elements commuting with $\rho(H)$.*

Proof. Any homomorphism $H \to G$ extends to $C(\rho) \times H \to G$, and since B preserves products, we get $BC(\rho) \times BH \to BG$, or

$$\coprod_{\rho \in \mathrm{Rep}(H,G)} BC(\rho) \to \mathrm{map}(BH, BG).$$

By the above, we can see that this is an isomorphism on π_0. Now fixing a basepoint ρ, we find that

$$[S^1, \mathrm{map}(BH, BG)]_* \cong [S^1 \times BH, BG]_{/BH},$$

where the decoration at the end means that we're looking at diagrams of the form

$$\begin{array}{ccc} S^1 \times BH & \longrightarrow & BG \\ \uparrow & \nearrow_{B\rho} & \\ * \times BH. & & \end{array}$$

These maps are determined up to homotopy by their effects on π_1, i. e., as elements of $\mathrm{Hom}(\mathbb{Z} \times H, G)_{/\rho}$. The '$/\rho$' determines the effect of such a map on H, so we only have to look at the map $\mathbb{Z} \to G$, which must land in $C(\rho)$. This ends the proof. □

If $H = \mathbb{Z}$, $BH = S^1$. This shows that the free loop space of BG, the space $\mathrm{map}(S^1, BG)$, is just $\coprod_{x \in G/\mathrm{conj}} BC(x)$.

Now suppose that G is a Lie group; in fact, let $G = U(n)$. (Note $U(1) = S^1$, and $BS^1 = \mathbb{C}P^\infty$.) The maximal torus is a map

$$T = \underbrace{S^1 \times \cdots \times S^1}_{n} \to U(n)$$

which on classifying spaces is

$$\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to BU(n).$$

The symmetric group Σ_n acts on the left-hand side, and on cohomology,

$$H^*BU(n) \cong H^*(\mathbb{C}P^{\infty, \times n})^{\Sigma_n} = \mathbb{F}_p[x_1, \ldots, x_n]^{\Sigma_n} \cong \mathbb{F}_p[c_1, \ldots, c_n].$$

where $|x_i| = 2$ and c_i are the universal Chern classes.

If H is discrete, then $\mathrm{Rep}(H, U(n))$ is the set of n-dimensional complex representations of H. We still have a map

$$\coprod_{\rho \in \mathrm{Rep}(H, U(n))} BC(\rho) \to \mathrm{map}(BH, BU(n)).$$

But without covering space theory, there's little hope for this map to be a weak equivalence.

Example 3. Let $H = C_p \cong \mathbb{Z}/p$. $\mathrm{Rep}(C_p, U(1))$ is multiplication by ζ^i, $0 \le i \le p-1$, where ζ is a primitive pth root of unity. That is, each i gives us a representation

$$\begin{gathered} \{1, \tau, \ldots, \tau^{p-1}\} = C_p \longrightarrow U(1) = S^1 \\ \rho_i : \tau \mapsto \zeta^i. \end{gathered}$$

Since $U(1)$ is abelian, $C(\rho_i) = U(1) = S^1$. Thus, the above map is just

$$\coprod_{i=0}^{p-1} \mathbb{C}P^\infty \to \mathrm{map}(B\mathbb{Z}/p, \mathbb{C}P^\infty).$$

Suppose we knew that $TH^*\mathbb{C}P^\infty = H^*\mathrm{map}(B\mathbb{Z}/p, \mathbb{C}P^\infty)$. The left-hand side is then

$$U(F(0)) \otimes U(\Phi(F(1))) \cong \mathbb{F}_p[x]/(x^p - x) \otimes H^*\mathbb{C}P^\infty.$$

But $x^p - x$ is a separable polynomial over $\mathbb{F}_p$, all of whose roots are in $\mathbb{F}_p$. Thus, we get

$$\mathbb{F}_p[x]/(x^p - x) \cong \prod_{i=0}^{p-1} \mathbb{F}_p \quad \text{via} \quad f(x) \mapsto (f(0), f(1), \ldots, f(p-1)).$$

Thus,

$$H^*\mathrm{map}(B\mathbb{Z}/p, \mathbb{C}P^\infty) \cong \left(\prod_{i=0}^{p-1} \mathbb{F}_p \right) \otimes H^*\mathbb{C}P^\infty \cong H^* \left(\coprod_{i=0}^{p-1} \mathbb{C}P^\infty \right).$$

Example 4. Again take $H = C_p = \mathbb{Z}/p$, and now $G = U(n)$. Every n-dimensional representation of C_p is diagonalizable, and thus of the form

$$\rho(\tau) = \begin{pmatrix} \zeta^{i_1} & 0 & \dots & 0 \\ 0 & \zeta^{i_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \zeta^{i_n} \end{pmatrix},$$

where without loss of generality, $i_1 \geq \cdots \geq i_n$. Write $(i_1, \dots, i_n) = (j_1, \dots, j_1, j_2, \dots, j_2, \dots, j_k)$ with $j_1 > \cdots > j_k$ and there are s_m copies of each j_m. Then

$$C(\rho) = U(s_1) \times \cdots \times U(s_k).$$

Thus,

$$\coprod_{\rho \in \mathrm{Rep}(C_p, U(n))} BC(\rho) = \coprod_{\rho} BU(s_1) \times \cdots \times BU(s_k).$$

If $n = 2$, there are just two cases: $i_1 = i_2$, or $i_1 > i_2$. So

$$\coprod_{\rho \in \mathrm{Rep}(C_p, U(2))} BC(\rho) = \coprod_{p-1 \geq i_1 = i_2 \geq 0} BU(2) \sqcup \coprod_{p-1 \geq i_1 > i_2 \geq 0} BU(1) \times BU(1).$$

What is $TH^*BU(2)$? Well, T is exact, so we can pull out the Σ_2-action: $TH^*BU(2) \cong T(H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty))^{\Sigma_2}$. T also commutes with tensor products, so this is

$$(TH^*\mathbb{C}P^\infty \otimes TH^*\mathbb{C}P^\infty)^{\Sigma_2} \cong [\mathbb{F}_p[y_1, y_2, x_1, x_2]/(x_i^p - x_i)]^{\Sigma_2},$$

where $|y_i| = 2$, $|x_i| = 0$, and the symmetric group action is the obvious thing, switching 1's and 2's. We can now calculate that

$$TH^*BU(2) = \prod_{i_1 = i_2} \mathbb{F}_p[y_1, y_2]_2^{\Sigma} x_1^{i_1} x_2^{i_2} \times \prod \mathbb{F}_p[y_1, y_2] x_1^{i_1} x_2^{i_2}.$$

The polynomials in the right-hand factor are not Σ_2-invariant, but they correspond bijectively to the Σ_2-invariants, via

$$f(y_1, y_2) x_1^{i_1} x_2^{i_2} \mapsto f(y_1, y_2) x_1^{i_1} x_2^{i_2} + f(y_2, y_1) x_1^{i_2} x_2^{i_1}.$$

And finally, the first factor is $p-1$ copies of $H^*BU(2)$, and the second is $\binom{p-1}{2}$ copies of $H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, exactly as predicted.

These are calculations for nontrivial spaces, that have given us very good evidence for the conjecture that $TH^*X = H^*\mathrm{map}(BC_p, X)$. As stated, this isn't quite true – for instance, you have to p-complete – but we're on the way to stating and proving the truth.