Lecture 13: The homotopy spectral sequence of a (co)simplicial space

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Last time, we introduced simplicial objects $\mathsf{s}\mathcal{C} = \{X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}\}$, and defined the skeleton filtration, in which $sk_nX \subseteq X$ was (morally) generated by the non-degenerate simplices in degree $\leq n$. There is a pushout diagram

$$\begin{array}{ccc} \partial\Delta^n \otimes X_n \coprod_{\partial\Delta^n \otimes L_nX} \Delta^n \otimes L_nX & \longrightarrow & sk_{n-1}X \\ \downarrow & & \downarrow \\ \Delta^n \otimes X & \longrightarrow & sk_nX \end{array}$$

saying that we can get the n-skeleton by attaching the nondegenerate n-simplices to the $(n-1)$-skeleton. Maybe it's worth writing down an example.

Definition 1. Let $\mathcal{C} = \mathsf{Top}$. The **geometric realization functor**

$$|\cdot| : \mathsf{sTop} \to \mathsf{Top}$$

is defined by the coequalizer diagram

$$\coprod_{\phi:[n]\to[m]} \Delta^n \times X_m \rightrightarrows \coprod_n \Delta^n \times X_n \longrightarrow |X|.$$

Here Δ^n is the topological n-simplex. The two maps are

$$\begin{array}{ccc} \Delta^n \times X_m & \xrightarrow{\phi \times X_m} & \Delta^m \times X_m \\ {\scriptstyle \Delta^n \times \phi^*}\downarrow & & \\ \Delta^n \times X_n. & & \end{array}$$

Theorem 2. *There are pushout diagrams in* Top

$$\begin{array}{ccc} \partial\Delta^n \times X_n \coprod_{\partial\Delta^n \times L_nX} \Delta^n \times L_nX & \longrightarrow & |sk_{n-1}X| \\ {\scriptstyle j_n}\downarrow & & \downarrow \\ \Delta^n \times X_n & \longrightarrow & |sk_nX|. \end{array}$$

(Recall that

$$L_nX = \operatorname{colim}_{\substack{\phi:[n]\twoheadrightarrow[m] \\ \phi \neq id}} X_m = \bigcup_{i=0}^{m-1} s_iX_{n-1}.)$$

Also, $\operatorname{colim} |sk_nX| = |X|$.

Suppose that $L_nX \to X_n$ is a cofibration (a relative CW-complex, if you'd like, or a neighborhood retract). Then define $NX_n = X_n/L_nX$. The cofiber of j_n is then $\Delta^n/\partial\Delta^n \wedge NX_n = \Sigma^n NX_n$.

Application: Let E_* be your favorite homology theory. We have this little picture

$$\begin{array}{ccccccc} |sk_0X| & \longrightarrow & |sk_1X| & \longrightarrow & |sk_2X| & \longrightarrow \cdots & |X| \\ \| & & \downarrow & & \downarrow & & \\ NX_0 & & \Sigma NX_1 & & \Sigma^2 NX_2 & & \end{array}$$

and applying E_* gives

$$\begin{array}{ccccccc} E_*|sk_0X| & \longrightarrow & E_*|sk_1X| & \longrightarrow & E_*|sk_2X| & \longrightarrow \cdots & E_*|X| \\ \| & & \downarrow & \nearrow & \downarrow & \nearrow & \\ E_*NX_0 & & E_*\Sigma NX_1 & & E_*\Sigma^2 NX_2 & & \end{array}$$

where the dotted arrows are connecting maps in long exact sequences. This can be rearranged into a spectral sequence

$$E^1_{s,t} = \widetilde{E}_{s+t}\Sigma^s NX_s = \widetilde{E}_t NX_s \Rightarrow E_{s+t}|X|.$$

Theorem 3 (Conservation of symbols)**.** *Under these hypotheses, $\widetilde{E}_t NX_s \cong N\widetilde{E}_t X_s$, the second N now meaning the normalization of the simplicial abelian group $\widetilde{E}_t X_\bullet$.*

Thus, $E^2_{s,t} = \pi_s \widetilde{E}_t X_\bullet = H_s(N\widetilde{E}_t, \partial)$.

Remark 4. Let $S_\bullet X$, for X a space, be given by $S_n X = \mathrm{map}(\Delta^n, X)$, and let $E_* = H(\cdot, \mathbb{Z})$. Then

$$E^2_{s,t} = \begin{cases} 0 & t \neq 0 \\ \pi_s N\mathbb{Z}S_\bullet X = H_s(\mathbb{Z}S_\bullet X, \partial). & \end{cases}$$

So $H_*|S_\bullet X| = H_* X$. This is most of the proof that $|S_\bullet X| \simeq X$.

Cosimplicial spaces

Let $X^\bullet$ be a cosimplicial space. The analogue of geometric realization is **totalization**, given as an equalizer

$$Tot(X) \longrightarrow \prod_n \mathrm{Hom}(\Delta^n, X^n) \rightrightarrows \prod_{\phi:[n]\to[m]} \mathrm{Hom}(\Delta^n, X^m).$$

Then we check that $\mathrm{Tot}(X) = \mathrm{map}_{\mathsf{sTop}}(\Delta^\bullet, X^\bullet)$.

We can also think of cosimplicial spaces as simplicial objects in $\mathsf{Top}^{\mathrm{op}}$, so they have a skeletal filtration, which in this context is called the **coskeletal filtration**. It gives a decomposition of Tot,

$$\mathrm{Tot}(X) \cong \lim \mathrm{Tot}_n X,$$

together with the pullback squares

$$\begin{array}{ccc} \mathrm{Tot}_n X & \longrightarrow & \mathrm{Hom}(\Delta^n, X^n) \\ {\scriptstyle p_n}\downarrow & \lrcorner & \downarrow{\scriptstyle j^n} \\ \mathrm{Tot}_{n-1} X & \longrightarrow & \mathrm{Hom}(\partial\Delta^n, X^n) \times_{\mathrm{Hom}(\partial\Delta^n, M^n X)} \mathrm{Hom}(\Delta^n, M^n X). \end{array}$$

Here $M^n X$ is the **matching space**

$$M^n X = \lim_{\substack{\phi:[n]\twoheadrightarrow[m] \\ \phi\neq id}} X^m.$$

As before, we can ask that $X^n \to M^n X$ is a fibration; then j^n is a fibration, and we get a tower of fibrations. Unfortunately, in order to start defining homotopy groups, we need to choose a basepoint, and this may not be possible. In fact, it can happen that $\mathrm{Tot}\, X = \emptyset$!

Assume that $X^n \to M^n X$ is a fibration, and let X^{-1} be the equalizer of $d_0, d_1 : X^0 \to X^1$. Assume $X^{-1} \neq \emptyset$. Then there are maps

$$X^{-1} = \mathrm{map}_{\mathsf{cTop}}(*, X^\bullet) \to \mathrm{map}_{\mathsf{cTop}}(\Delta^\bullet, X^\bullet) = \mathrm{Tot}(X),$$

and a choice of basepoint for X^{-1} is a choice of basepoint for the Tot tower (necessarily a constant map $\Delta^\bullet t\o X^\bullet$).

If $N^n X$ is the fiber of $X^n \to M^n X$ at this basepoint, then the fiber of j^n is $\Omega^n N^n X$, and you get a second quadrant spectral sequence

$$H^s(N\pi_t X^\bullet, \partial) = \pi^s \pi_t X^\bullet \Rightarrow \pi_{t-s} \mathrm{Tot}(X).$$

(Careful! When $t = 0$, the left-hand side only makes sense for $s = 0$; when $t = 1$, only for $s = 0$ or 1.)

There's much more to discuss: when does this converge? What's with the restrictions on s? What if you can't choose a basepoint? We'll get to this next class.