Lecture 14: The Bousfield-Kan spectral sequence

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Last time, we defined the Bousfield-Kan spectral sequence, which was an example of a homotopy spectral sequence of a cosimplicial space.

Let's go back to algebra. Recall that $\mathcal{K}$ is the category of unstable algebras. Consider the functor $G : \mathsf{GradedVectorSpaces} \to \mathcal{K}$ with $G(V) = H^*(K(V), \mathbb{F}_p)$ (if V is finite type). If $V = \Sigma^n \mathbb{F}_p$ (i. e. living in degree n), then $H^* G(\Sigma^n \mathbb{F}_p) = U(F(n)) = H^* K(\mathbb{Z}/p, n)$.

Lemma 1. *Let X be a space of finite type. Then there is a functor $\mathbb{F}_p(\bullet) : \mathsf{Spaces} \to \mathsf{Spaces}$ such that there is a natural isomorphism*

$$H^* \mathbb{F}_p(X) \cong G(H^* X).$$

Remark 2. Non-canonically,

$$\mathbb{F}_p(X) = \prod_{n \geq 0} K(H_n X, n).$$

In fact, this is already functorial as a homotopy type, but we need it to be functorial as a space.

Remark 3. If X is a simplicial set, then $\mathbb{F}_p(X)$ is the simplicial $\mathbb{F}_p$-vector space in X. If X is a CW-complex, we can define $\mathbb{F}_p(X) = \Omega^\infty (H\mathbb{F}_p \wedge X_+)$.

This functor could be used to assemble a cosimplicial space

$$X \longrightarrow \mathbb{F}_p X \rightrightarrows \mathbb{F}_p^2 X \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots .$$

More formally, $\mathbb{F}_p(\cdot)$ is part of a **monad**, i. e., there are maps $\epsilon : X \to \mathbb{F}_p X$ and $\mathbb{F}_p^2 X \to \mathbb{F}_p X$ inducing the above cosimplicial diagram. Thinking in terms of (simplicial) sets, the first map is just inclusion of the basis of a (simplicial) vector space, and the second map is addition inside a (simplicial) vector space.

Proposition 4. *If X is of finite type, then*

$$H^* \Big(X \longrightarrow \mathbb{F}_p X \rightrightarrows \mathbb{F}_p^2 X \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots \Big)$$

is isomorphic to

$$H^* X \longleftarrow G(H^* X) \rightleftarrows G^2(H^* X) \mathrel{\substack{\rightarrow\\\leftarrow\\\rightarrow}} \cdots$$

because $H^* \mathbb{F}_p X \cong G(H^* X)$. This is the canonical $\mathcal{K}$-resolution of $H^* X$.

Sidebar

Let X and Y be spaces, and let $f : X \to Y$ be a continuous map. Consider a map $\alpha : S^t \to \mathrm{map}(X, Y)$ sending the basepoint to f; we can think of this as an element of $\pi_t(\mathrm{map}(X, Y); f)$.

The adjoint of this is a map

$$\begin{array}{ccc} S^t \times X & \xrightarrow{\beta} & Y \\ \uparrow & \nearrow_{f} & \\ * \times X & & \end{array}$$

where $\beta(a,x) = \alpha(a)(x)$ (and recalling $\alpha(*)(x) = f(x)$). Taking cohomology gives

$$\begin{array}{ccc} H^*(S^t \times X) & \xleftarrow{\beta^*} & H^*Y \\ \downarrow & \swarrow_{f^*} & \\ H^*X. & & \end{array}$$

But $H^*(S^t \times X) \cong E(x_t) \otimes H^*X$, so we can write

$$\beta^*(y) = 1 \otimes f^*(y) + x_t \otimes \partial(y).$$

β^* is a ring map, so given y and z, we get

$$\begin{aligned} 1 \otimes f^*(yz) + x_t \otimes \partial(yz) &= (1 \otimes f^*y + x_t \otimes partialy)(1 \otimes f^*z + x_t \otimes \partial z) \\ &= 1 \otimes f^*(yz) + x_t \otimes f^*y(\partial z) + x_t \otimes (\partial y)f^*z \end{aligned}$$

using that f^* is a ring map and that $x_t^2 = 0$. We conclude that

$$\partial(yz) = f^*y\partial z + \partial y f^*z = y\partial z + (\partial y)z$$

(suppressing the f^*s in the last formula). That is, ∂ is a derivation from H^*Y to H^*X, where we consider H^*X as an H^*Y-module via f^*. We thus have a Hurewicz map

$$h : \pi_t(\mathrm{map}(X,Y); f) \to \mathrm{Der}_{\mathcal{K}}(H^*Y, \Sigma^t H^*X; f^*).$$

The subscript $\mathcal{K}$ is because these derivations also commute with Steenrod operations (because these are all trivial on the sphere), and the f^* shows the module structure of $\Sigma^t H^*X$.

Proposition 5. *If Y is of finite type and a product of $K(\mathbb{Z}/p,n)$s (a* ***generalized Eilenberg-Mac Lane space****), then h is an isomorphism.*

Proof. If X is any space of finite type, and Y is generalized Eilenberg-Mac Lane space, then we have to show

$$[X,Y] \cong \mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*X).$$

The basic case is $Y = K(\mathbb{Z}/p,n)$. Then

$$H^nX = [X, K(\mathbb{Z}/p,n)] = \mathrm{Hom}_{\mathcal{K}}(H^*K(\mathbb{Z}/p,n), H^*X) = H^nX,$$

the last step since $H^*K(\mathbb{Z}/p,n) = U(F(n))$. The other generalized Eilenberg-Mac Lane spaces are products of these, so the theorem follows for them too. To get the theorem in higher degrees, use

$$\mathrm{Der}_{\mathcal{K}}(H^*Y, H^*\Sigma^t X) \cong \mathrm{Hom}_{\mathcal{K}/H^*X}(H^*X, H^*(S^t \times X)).$$

□

Let $f : X \to Y$ be a map; consider the augmented cosimplicial space

$$\mathrm{map}(X,Y) \longrightarrow \mathrm{map}(X, \mathbb{F}_pY) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \mathrm{map}(X, \mathbb{F}_p^2Y) \qquad \cdots$$

with basepoint $f \in \mathrm{map}(X,Y)$. We get a spectral sequence

$$\pi^s\pi_t\mathrm{map}(X, \mathbb{F}_p^\bullet Y) \Rightarrow \pi_{t-s}\mathrm{map}(X, \mathrm{Tot}(\mathbb{F}_p^\bullet Y)).$$

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$$\pi^s\pi_t\mathrm{map}(X, \mathbb{F}_p^\bullet Y) \cong \pi^s\mathrm{Der}_{\mathcal{K}}(H^*\mathbb{F}_p^\bullet Y, \Sigma^t H^*X)$$

since Y is a generalized Eilenberg-Mac Lane space, which is

$$\pi^s\mathrm{Der}_{\mathcal{K}}(G^\bullet(H^*Y), \Sigma^t H^*X) \cong R^s\mathrm{Der}_{\mathcal{K}}(H^*Y, \Sigma^t H^*X)_{f^*}.$$

The $G^\bullet$ is a simplicial resolution in $\mathcal{K}$, so this is one way to define the right derived functors, if you want.

Definition 6. The map $X \to \mathrm{Tot}(F_p^\bullet X) =: (\mathbb{F}_p)_\infty X$ is the **Bousfield-Kan p-completion** of X.

Theorem 7. *If X is 1-connected and finite type, then*

$$\pi_t(\mathbb{F}_p)_\infty X = (\pi_t X)_p^\wedge.$$

For example, let's take $X = *$. Then $\mathrm{map}(X, Z) = Z$. If Y is path-connected, the spectral sequence is

$$R^s\mathrm{Der}_{\mathcal{K}}(H^*Y, \Sigma^t\mathbb{F}_p) \Rightarrow \pi_{t-s}(\mathbb{F}_p)_\infty Y.$$

Suppose also that $H^*Y = U(M)$ for some M. For example, $H^*S^t = U(\Sigma^t\mathbb{F}_p)$.

Lemma 8. $R^s\mathrm{Der}_{\mathcal{K}}(H^*Y, \Sigma^t\mathbb{F}_p) \cong \mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma^t\mathbb{F}_p)$.

This is a more classical version of the Bousfield-Kan spectral sequence: a spectral sequence

$$\mathrm{Ext}^s_{\mathcal{U}}(\Sigma^n\mathbb{F}_p, \Sigma^t\mathbb{F}_p) \Rightarrow \pi_{t-s}(S^n)_p^\wedge.$$

The Adams spectral sequence is just a stable version of this:

$$\mathrm{Ext}^s_{\mathcal{A}}(\mathbb{F}_p, \Sigma^t\mathbb{F}_p) \Rightarrow (\pi^{\mathrm{st}}_{t-s} S^0)_p^\wedge.$$