Lecture 15: Computations with the Bousfield-Kan spectral sequence

November 3, 2014

No class on Friday!

Last time, we introduced the Bousfield-Kan spectral sequence. Given a continuous map of spaces (or a map of simplicial sets $f : X \to Y$), this is a spectral sequence

$$R^s \operatorname{Der}_{\mathcal{K}}(H^*Y, \Sigma^t H^* X)_{f^*} \Rightarrow \pi_{t-s}(\operatorname{map}(X, (\mathbb{F}_p)_\infty(Y)); f),$$

where $(\mathbb{F}_p)_\infty(Y)$ is the Bousfield-Kan p-completion.

Example 1. Let $X = *$ and let Y be simply connected. Then $\operatorname{map}(X, \mathbb{F}_{p^\infty} Y) \cong Y$, and derivations into $\Sigma^t \mathbb{F}_p$ are the same as homomorphisms into it, so we get

$$R^s \operatorname{Der}_{\mathcal{K}}(H^*Y, \Sigma^t \mathbb{F}_p) \cong \operatorname{Ext}^s_{\mathcal{K}}(H^*Y, H^*S^t) \Rightarrow \pi_{t-s}(\mathbb{F}_p)_\infty Y.$$

This is the **unstable Adams spectral sequence**.

Proposition 2. *Suppose $K = U(M)$ and $M^0 = 0$. Then*

$$\operatorname{Ext}^s_{\mathcal{K}}(U(M), H^*S^t) \cong \operatorname{Ext}^s_{\mathcal{U}}(M, \Sigma^t \mathbb{F}_p).$$

For example, this gives

$$\operatorname{Ext}^s_{\mathcal{K}}(H^*S^n, H^*S^t) \cong \operatorname{Ext}^s_{\mathcal{U}}(\Sigma^n \mathbb{F}_p, \Sigma^t \mathbb{F}_p).$$

Proof. Choose a simplicial projective resolution in $\mathcal{U}$ of M,

$$\cdots \qquad P_1 \rightrightarrows P_0 \longrightarrow M$$

(the arrows along the bottom are a simplicial contraction, which is part of the definition of a resolution here). Here's what the simplicial stuff buys you: applying U preserves all the simplicial relations and the relations making the bottom arrows a contraction, so $U(P_\bullet)$ is also a simplicial resolution of $U(M)$, this time in $\mathcal{K}$. So we get

$$\operatorname{Ext}^s_{\mathcal{U}}(U(M), H^*S^t) = \pi^S \operatorname{Hom}_{\mathcal{K}}(U(P_\bullet), H^*S^t) \cong \pi^S \operatorname{Hom}_{\mathcal{U}}(P_\bullet, \Sigma^t \mathbb{F}_p).$$

The last step is using adjoint functors and the fact that $M^0 = 0$, so no maps land in $H^0 S^t$. Thus, we get

$$\operatorname{Ext}^s_{\mathcal{K}}(U(M), H^*S^t) \cong \operatorname{Ext}^s_{\mathcal{U}}(\Sigma^t \mathbb{F}_p).$$

□

Recall that there's an adjunction in $\mathcal{U}$,

$$\operatorname{Hom}_{\mathcal{U}}(M, \Sigma N) \cong \operatorname{Hom}_{\mathcal{U}}(\Omega M, N).$$

Proposition 3. *For all s there is a spectral sequence*

$$\operatorname{Ext}^p_{\mathcal{U}}(\Omega^s_q M, N) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{U}}(M, \Sigma^s N).$$

Proof. This is a standard composite-functor spectral sequence, of which you can find constructions elsewhere. We might go over one in detail when we're in a less abelian setting. □

Example 4. Let $s = 1$. The spectral sequence is now just

$$\mathrm{Ext}^p_{\mathcal{U}}(\Omega_q M, N) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{U}}(M, \Sigma N).$$

We also know that $\Omega_q M = 0$ for $q > 1$, so there's only one differential – the spectral sequencce is really just a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^s_{\mathcal{U}}(\Omega M, N) \longrightarrow \mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma N) \longrightarrow \mathrm{Ext}^{s-1}_{\mathcal{U}}(\Omega_1 M, N) \xrightarrow{d_2} \mathrm{Ext}^{s+1}_{\mathcal{U}}(\Omega M, N) \longrightarrow \cdots.$$

Let's specialize even further: let's take $p = 2$ and $M = \Sigma^{n+1}\mathbb{F}_2$, $N = \Sigma^t \mathbb{F}_2$. The long exact sequence is

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{Ext}^s_{\mathcal{U}}(\Sigma^n \mathbb{F}_2, \Sigma^t \mathbb{F}_2) & \xrightarrow{E} & \mathrm{Ext}^s_{\mathcal{U}}(\Sigma^{n+1}\mathbb{F}_2, \Sigma^{t+1}\mathbb{F}_2) & \xrightarrow{H} & \mathrm{Ext}^{s-1}_{\mathcal{U}}(\Sigma^{2n+1}\mathbb{F}_2, \Sigma^t \mathbb{F}_2) & \xrightarrow{P} & \mathrm{Ext}^{s+1}_{\mathcal{U}}(\Sigma^n \mathbb{F}_2, \Sigma^t \mathbb{F}_2) & \longrightarrow \cdots \\ & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \\ \cdots \longrightarrow & \pi_{t-s} S^n & \xrightarrow{E} & \pi_{t+1-s} S^{n+1} & \xrightarrow{H} & \pi_{t+1-s} S^{2n+1} & \xrightarrow{P} & \pi_{t-s-1} S^n & \longrightarrow \cdots. \end{array}$$

The sequence in homotopy is a famous long exact sequence called the **EHP sequence**: E for *Einhängung*, the German word for 'suspension', H for Hopf, and P for Whitehead product. On homotopy, it's induced by a 2-local fibration

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.$$

There are a few things we can say about this. Since $(\Sigma^n \mathbb{F}_2)^k = 0$ for $k < n$, we can show that $\mathrm{Ext}^s_{\mathcal{U}}(\Sigma^n \mathbb{F}_2, \Sigma^t \mathbb{F}_2) = 0$ for $t - s < n$. In particular, the term with the Σ^{2n+1} has this range decreasing by twos, so $E : \mathrm{Ext}^s_{\mathcal{U}}(\Sigma^n \mathbb{F}_2, \Sigma^t \mathbb{F}_2) \to \mathrm{Ext}^s_{\mathcal{U}}(\Sigma^{n+1}\mathbb{F}_2, \Sigma^{t+1}\mathbb{F}_2)$ is an isomorphism a lot of the time, namely when $t - s \leq 2n - 1$ (the stable range). Its stable value is the standard Ext term over the Steenrod algebra.

Secondly,

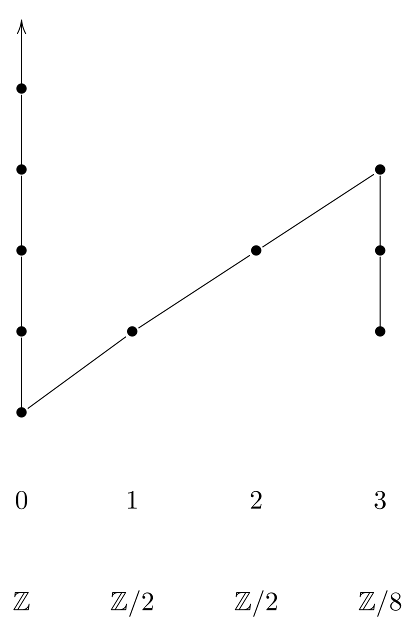
$$\mathrm{Ext}^s_{\mathcal{U}}(\Sigma \mathbb{F}_2, \Sigma^t \mathbb{F}_2) = \begin{cases} 0 & t - s \neq 1 \\ \mathbb{F}_2 & t - s = 1. \end{cases}$$

This converges to (and is) the homotopy groups of the circle – or at least their associated graded, filtering $\pi_1 S^1 = \mathbb{Z}$ by the 2-adic filtration. We can prove this by using the projective resolution

$$\cdots \longrightarrow F(4) \xrightarrow{\mathrm{Sq}^1} F(3) \xrightarrow{\mathrm{Sq}^1} F(2) \xrightarrow{\mathrm{Sq}^1} F(1) \longrightarrow \Sigma \mathbb{F}_2.$$

Thus, $\pi_1(\mathbb{F}_2)_\infty S^1 \cong \mathbb{Z}_2$.

Let's now try to do this for S^2. We know what the answer is, in a small range. The standard way of drawing the stable Ext is in a chart with $t - s$ drawn horizontally and s drawn vertically. In low degrees, this looks like



Here the vertical lines denote multiplication by 2, and the diagonal lines multiplication by η, the generator of π_1.

We have

$$\mathrm{Ext}^s_{\mathcal{U}}(\Sigma^2\mathbb{F}_2, \Sigma^t\mathbb{F}_2) \cong \mathrm{Ext}^s_{\mathcal{U}}(\Sigma^3\mathbb{F}_2, \Sigma^{t+1}\mathbb{F}_2)$$

for $t - s = 3$.

(At this point, PG used the EHP sequences for S^2, S^3, and S^4, and knowledge of the above cohomology groups, to very rapidly calculate the homotopy groups of these spheres in low degrees, and your intrepid scribe was unable to keep up.)