Lecture 16: Towers of fibrations

November 5, 2014

The homotopy spectral sequence of a tower of fibrations

Suppose given a pullback square

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow f \\ X & \xrightarrow[g]{} & B \end{array}$$

with f a fibration, and let $* \in (X \times_B Y)$ be a basepoint. There's a long exact sequence in homotopy dual to the Mayer-Vietoris sequence. If $n \geq 2$, this is an exact sequence

$$\cdots \to \pi_n(X \times_B Y) \to \pi_n X \times \pi_n Y \stackrel{f_* - g_*}{\to} \pi_n B \to \pi_{n-1}(X \times_Y B) \to \cdots .$$

If $n = 1$, it's a sequence

$$\pi_2 B \longrightarrow \pi_1(X \times_B Y) \longrightarrow \pi_1 X \times \pi_1 Y \xrightarrow{f_* g_*^{-1}} \pi_1 B \longrightarrow \pi_0(X \times_B Y) \longrightarrow \pi_0 X \times \pi_0 Y \longrightarrow \pi_0 B.$$

What does exactness mean here? It makes sense at $\pi_1(X \times_B Y)$. At $\pi_0(X \times_B Y)$, it means that $\pi_1 B$ acts on $\pi_1(X \times_B Y)$ with orbit space the pullback $\pi_0 X \times_{\pi_0 B} \pi_0 Y$, and the isotropy subgroup is the image of $\pi_1 X \times \pi_1 Y$ in $\pi_1 B$. N(ote that $f_* g_*^{-1}$ isn't a group homomorphism!) This is proven by judicious use of path-lifting properties.

Now let's consider a tower of fibrations (all pointed)

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ F_3 & \longrightarrow & X_3 \\ & & \downarrow \\ F_2 & \longrightarrow & X_2 \\ & & \downarrow \\ F_1 & \longrightarrow & X_1 \\ & & \downarrow \\ & & X_0 \end{array}$$

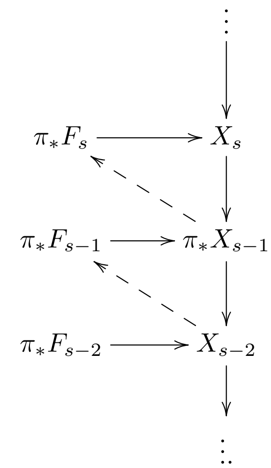
with the following extra structure: a pullback square

$$\begin{array}{ccc} X_n & \longrightarrow & E_n \\ \downarrow & \lrcorner & \downarrow p_n \\ X_{n-1} & \longrightarrow & B_n \end{array}$$

where the fiber of p_n is F_n. (Note that every tower of fibrations admits trivial examples of this structure, with $E_n = X_n$ and $B_n = X_{n-1}$, but we're interested in more interesting cases – in particular, when B_n is an Eilenberg-Mac Lane space.) We get a spectral sequence

$$E_1^{s,t} = \pi_{t-s} F_s \Rightarrow \pi_{t-s} \lim X_s,$$

induced by the exact couple



The dotted arrows composed with the horizontal arrows are the d_1s, and in general, d_r is a map $E_r^{s,t} \to E_r^{s+r,t+r-1}$. We can draw this as a second-quadrant spectral sequence, with s the horizontal axis, t the vertical axis, and the differentials going up and to the left. Again, we have to be careful in low homotopy degrees: the extra structure of pullback squares gives us a differential out of π_0. (How?)

Let's discuss convergence. $E_\infty^{s,t}$ is a subquotient of $\lim_s \pi_{t-s} X_s$. We say that the spectral sequence **converges** if

$$\pi_{t-s} \lim X_s \xrightarrow{\sim} \lim \pi_{t-s} X_s.$$

This needn't happen: in general, there's a $\lim^1$ term in the kernel.

Proposition 1. *A useful criterion for convergence is the following: the spectral sequence converges if for all n there is an r such that*

$$E_r^{s,t} = E_\infty^{s,t} \quad \text{for } t - s = n.$$

Example 2. Let X be a pointed space, and consider the Bousfield-Kan resolution

$$X \longrightarrow \mathbb{F}_p(X) \rightrightarrows \mathbb{F}_p^2(X) \rightrightarrows \cdots .$$

(Recall that $\mathbb{F}_p(X)$ was a product of Eilenberg-Mac Lane spaces with the property that $\pi_* \mathbb{F}_p(X) \cong H_*(X; \mathbb{F}_p)$; we can take $\mathbb{F}_p = \Omega^\infty (H\mathbb{F}_p \wedge \Sigma^\infty X_+)$. From this cosimplicial space we get a Tot tower of fibrations, with pullback squares

$$\begin{array}{ccc} \mathrm{Tot}_n \, \mathbb{F}_p^\bullet(X) & \longrightarrow & \mathrm{map}(\Delta^n, \mathbb{F}_p^{n+1} X) \\ \downarrow & & \downarrow \\ \mathrm{Tot}_{n-1} \, \mathbb{F}_p^\bullet(X) & \longrightarrow & \mathrm{map}(\partial\Delta^n, \mathbb{F}_p^{n+1} X) \times_{\mathrm{map}} (\partial\Delta^n, M^n \mathbb{F}_p^\bullet X) \mathrm{map}(\Delta^n, M_n \mathbb{F}_p^\bullet X). \end{array}$$

In fact, this diagram simplifies considerably, since the coface and codegeneracy maps other than d_0 are all $\mathbb{F}_p(\bullet)$. So the fibration

$$N_n \mathbb{F}_p^\bullet X \longrightarrow \mathbb{F}_p^{n+1} X \longrightarrow M_n \mathbb{F}_p^\bullet X$$

splits, and the middle space is a product (this is essentially the Dold-Kan theorem). So the pullback diagram simplifies to

$$\begin{array}{ccc}
\Omega_n N_n \mathbb{F}_p^\bullet X & = & \mathrm{map}_*(\Delta^n/\partial\Delta^n, N_n\mathbb{F}_p^\bullet(X)) \\
\downarrow & & \downarrow \\
\mathrm{Tot}_n\, \mathbb{F}_p^\bullet(X) & \longrightarrow & \mathrm{map}(\Delta^n, N_n\mathbb{F}_p^\bullet(X)) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{Tot}_{n-1}\, \mathbb{F}_p^\bullet(X) & \longrightarrow & \mathrm{map}(\partial\Delta^n, N_n\mathbb{F}_p^\bullet(X)).
\end{array}$$

Letting $F_n = \Omega_n N_n \mathbb{F}_p^\bullet X$, we get $\pi_{t-s}F_s = \pi_t N_s \mathbb{F}_p^\bullet X = N_s \pi_t \mathbb{F}_p^\bullet X$ – this s-fold loops functor is what gives the shift in homotopy at the end of the spectral sequence.

Here's a variation on this example: use the resolution

$$\mathrm{map}(Y, X) \to \mathrm{map}(Y, \mathbb{F}_p^\bullet X).$$

Since Tot is maps out of something, it can be pulled out of maps into things. So the Tot tower gives

$$\begin{array}{ccc}
\mathrm{map}(Y, \mathrm{Tot}_n\, X) & \longrightarrow & \mathrm{map}(Y \times \Delta^n, N_n\mathbb{F}_p^\bullet X) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{map}(Y, \mathrm{Tot}_{n-1}\, X) & \longrightarrow & \mathrm{map}(Y \times \partial\Delta^n, N_n\mathbb{F}_p^\bullet X).
\end{array}$$

In degrees 0 and 1, this is

$$\begin{array}{ccc}
\mathrm{map}(Y, \mathrm{Tot}_1\, \mathbb{F}_p^\bullet X) & \longrightarrow & \mathrm{map}(Y \times \Delta^1, N_1\mathbb{F}_p^\bullet X) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{map}(Y, \mathbb{F}_p^\bullet X) & \longrightarrow & \mathrm{map}(Y \times \partial\Delta^1, N_1\mathbb{F}_p^\bullet X).
\end{array}$$

Given $f \in \pi_0\mathrm{map}(Y, \mathbb{F}_p X) = \mathrm{Hom}_{\mathcal{K}}(H^*\mathbb{F}_p X, H^*Y)$ (assuming that X and Y are finite type), f lifts to $\pi_0\mathrm{map}(Y, \mathrm{Tot}_1\, \mathbb{F}_p^\bullet X)$, and we then get successive obstructions to lifting this up the Tot tower. These obstructions were analyzed by Bousfield in the early 80s.

Example 3. Let X be a path-connected pointed space, and Y a CW-complex. Then $\mathrm{map}(Y, P_\bullet X)$, where $P_\bullet X$ is the Postnikov tower, is such a tower of fibrations. Recall that the Postnikov tower is a functorial tower of fibrations

$$X \to \cdots \to P_n X \to P_{n-1}X \to \cdots \to P_1 X,$$

where $\pi_t X \to \pi_t P_n X$ is an isomorphism for $t \leq n$, $\pi_t P_n X = 0$ for $t > n$ (so $P_1 X = B\pi_1 X = B\pi$), and there's a homotopy pullback diagram of spaces over $B\pi$

$$\begin{array}{ccc}
P_n X & \longrightarrow & B\pi \\
\downarrow & & \downarrow \\
P_{n-1}X & \longrightarrow & E\pi \times_\pi K(\pi_n, n+1).
\end{array}$$

Here the map $B\pi \to E\pi \times_\pi K(\pi_n, n+1)$ is given by the inclusion of the basepoint to $K(\pi_n, n+1)$, which is fixed by π.

Here's an extremely rapid construction of the Postnikov tower, which you won't find in the blue book. Define $P_n^1 X$ by the pushout

$$\begin{array}{ccc}
\bigvee_{k>n} \bigvee_{f:S^k\to X} S^k & \longrightarrow & X \\
\downarrow & & \downarrow \\
\bigvee_{k>n} \bigvee_{f:S^k\to X} D^{k+1} & \longrightarrow & \ulcorner\, P_n^1 X,
\end{array}$$

and define $P_n X = \operatorname{colim}_i P_n^i X$. We're just coning off all the higher-dimensional spheres, rinsing, and repeating.

Define C by the homotopy pushout

$$\begin{array}{ccc} P_n X & \longrightarrow & P_1 X \simeq B\pi \\ \downarrow & & \downarrow \\ P_{n-1} X & \longrightarrow & C. \end{array}$$

By the Blakers-Massey excision theorem, there's a non-canonical isomorphism $P_{n+1} C = E\pi \times_\pi K(\pi_n, n+1)$, and one can check that after applying P_{n+1} to C, the above square is then a homotopy pullback as well.