Lecture 17: Approximating mapping spaces with the T-functor

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Recall that $T_V : \mathcal{K} \to \mathcal{K}$ (or $\mathcal{U} \to \mathcal{U}$) is an exact functor satisfying

$$\mathrm{Hom}_{\mathcal{K}}(T_V K, L) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*BV \otimes L).$$

$T_V H^*Y$ is an algebraic model for $H^* \mathrm{map}(BV, Y)$.

If Y is a space of finite type, then there's an evaluation map

$$\mathrm{map}(BV, Y) \times BV \to Y$$

inducing

$$H^* \mathrm{map}(BV, Y) \otimes H^*BV \leftarrow H^*Y,$$

which has an adjoint

$$H^* \mathrm{map}(BV, Y) \leftarrow T_V H^*Y.$$

Definition 1. Suppose H^*Y is finite type. Then (Z, ω) is a **model** for $\mathrm{map}(BV, Y)$ if

1. Z is a space with H^*Z isof finite type, and
2. $\omega : BV \times Z \to Y$ is a map so that $\widetilde{\omega} : T_V H^*Y \to H^*Z$ is an isomorphism.

Theorem 2 (Lannes)**.** *If (Z, ω) is a model for $\mathrm{map}(BV, Y)$, then there is a weak equivalence*

$$(\mathbb{F}_p)_\infty Z \xrightarrow{\sim} \mathrm{map}(BV, (\mathbb{F}_p)_\infty Y).$$

Remark 3. We have to do this p-completion because the model can only see cohomology. In special cases, though, we'll be able to remove it.

Example 4. Let Y be any space. Then

$$c : Y \to \mathrm{map}(BV, Y),$$

sending $y \in Y$ to the constant map to y, is adjoint to

$$p_2 : BV \times Y \to Y,$$

the projection onto the second factor. The algebraic analogue starts with

$$H^*BV \otimes H^*Y \leftarrow H^*Y : (p_2)^*;$$

this has an adjoint

$$H^*Y \leftarrow T_V H^*Y.$$

Theorem 5 (Miller)**.** *Let Y be a finite CW-complex. Then (Y, p_2) is a model for $\mathrm{map}(BV, Y)$. Hence $(\mathbb{F}_p)_\infty Y \simeq \mathrm{map}(BV, (\mathbb{F}_p)_\infty Y)$.*

We'll later show that we can get rid of the completions.

Proof. We just need to check that the adjoint map

$$T_V H^*Y \to H^*Y$$

to $(p_2)^*$ is an isomorphism. For $L \in \mathcal{U}$,

$$\mathrm{Hom}_{\mathcal{U}}(T_V H^*Y, L) \cong \mathrm{Hom}(H^*Y, H^*BV \otimes L).$$

But $H^*BV \cong \mathbb{F}_p \oplus \widetilde{H}^*BV$, and $\widetilde{H}^*BV$ is reduced, i. e. there are no nonzero Steenrod operations on it. Thus, there are no maps $H^*Y \to \widetilde{H}^*BV \otimes L$, and we get

$$\mathrm{Hom}_{\mathcal{U}}(H^*Y, H^*BV \otimes L) \cong \mathrm{Hom}_{\mathcal{U}}(H^*Y, L).$$

Thus, H^*Y and $T_V H^*Y$ is actually isomorphic. To find the isomorphism, we just have to be a little more careful, noting the actual isomorphisms between the Hom sets above, and use the functoriality of Yoneda's lemma: the isomorphism is precisely the adjoint to $(p_2)^*$. □

Example 6. Let $Y = BU(n)$. Then we constructed a map

$$BV \times \left(\coprod_{\rho \in \mathrm{Rep}(V, U(n))} BC(\rho) \right) = \coprod_{\rho \in \mathrm{Rep}(V, U(n))} BV \times BC(\rho) \xrightarrow{e} BU(n),$$

where $C(\rho)$ is the centralizer of ρ. Let $Z = \coprod_\rho BC(\rho)$, and $\omega = e$.

Theorem 7 (Dwyer-Zabrodsky)**.** *(Z, ω) is a model for* $\mathrm{map}(BV, BU(n))$*. Hence*

$$\coprod_\rho (\mathbb{F}_p)_\infty BC(\rho) \xrightarrow{\sim} \mathrm{map}(BV, (\mathbb{F}_p)_\infty BU(n)).$$

Again, we can get rid of the p-completion on the right-hand side, because BV only sees the p-completion of a simply connected space anyway. It's harder on the left-hand side.

Proof. We need to show that the adjoint of e^* gives an isomorphism

$$\bigoplus H^*BC(\rho) \cong T_V H^*BU(n).$$

This was done earlier, using invariant theory. □

In fact, the proof can be done for a more general compact group, just by embedding it into $U(n)$.

Now let's prove Lannes' theorem. We want to show that a map between two inverse limits is a weak equivalence. The following will give us a way to do this based on data from the E_2 page of the Bousfield-Kan spectral sequence.

Theorem 8 (Bousfield-Kan comparison lemma)**.** *Let $\{X_n\}, \{Y_n\}$ be pointed towers of fibrations such that $E_2^{s,s} = *$ for all $s > 0$ (for instance, if X_n and all F_n are path-connected, as in the Bousfield-Kan tower of a path-connected space.) Now let $f : \{X_n\} \to \{Y_n\}$ be a map of towers of fibrations so that*

$$f_* : E_2^{s,t}\{X_n\} \xrightarrow{\sim} E_2^{s,t}\{Y_n\}$$

for $t - s \geq 0$.

Proof. This can be found on p. 261 of Bousfield-Kan's yellow book. Let Z_n be the homotopy fiber of $X_n \to Y_n$ (possible since everything's pointed). This is also a pointed tower of fibrations, and $E_2^{s,t}\{Z_n\} = 0$. So the homotopy spectral sequence for $\{Z_n\}$ converges (to 0), and $\mathrm{holim}\, Z_n = *$. There's a fiber sequence

$$\mathrm{holim}\, Z_n \to \mathrm{holim}\, X_n \to \mathrm{holim}\, Y_n$$

and we last need to check that X_n and Y_n are both path-connected. This is implied by the hypotheses on the tower. □

In Lannes' theorem, $\{X_n\} = \{\mathrm{Tot}_n(\mathbb{F}_p^{\bullet} Z_\phi)\}$ where $Z_\phi \subseteq Z$ is a path component, and $\{Y_n\} = \{\mathrm{Tot}_n \operatorname{map}(BV, \mathbb{F}_p^{\bullet} Y)_\phi\}$. The first step, then, is to learn how to find path components.

Remark 9. Let's take a break and do a little algebra. Let A be a finite $\mathbb{F}_p$-algebra, so that $x^p = x$ for all $x \in A$ (such as $H^0(X; \mathbb{F}_p)$). Then

$$A \cong \prod_{\mathrm{Hom}_{\mathrm{alg}}(A,\mathbb{F}_p)} = \mathbb{F}_p(\mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{F}_p)) = \mathbb{F}_p^{\mathrm{Spec}(A)}.$$

Indeed, let $I \subsetneq A$ be a prime ideal. Then $\mathbb{F}_p \to A/I$ makes A/I a finite-dimensional integral domain over $\mathbb{F}_p$ in which $x^p = x$. But every element of A/I satisfies some minimal polynomial which divides $x^p - x$, so is in $\mathbb{F}_p$ already, so $A/I \cong \mathbb{F}_p$. Thus I is maximal with residue field $\mathbb{F}_p$.

Let $K \in \mathcal{K}$; then

$$\mathrm{Hom}_{\mathbb{F}_p-\mathrm{alg}}((T_V K)^0, \mathbb{F}_p) \cong \mathrm{Hom}_{\mathcal{K}}(T_V K, \mathbb{F}_p) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^* BV).$$

By the algebraic remark above, we get

$$(T_V K)^0 \cong \mathbb{F}_p^{\mathrm{Hom}_{\mathcal{K}}(K, H^* BV)}.$$

If Y is a space of finite type, then

$$H_0 Y = \mathbb{F}_p[\pi_0 Y]$$

and

$$H^0 Y = \mathbb{F}_p(\pi_0 Y);$$

in other words,

$$\pi_0 Y = \mathrm{Hom}_{\mathbb{F}_p-\mathrm{alg}}(H^0 Y, \mathbb{F}_p).$$

Proposition 10. *If (Z, ω) is a model for $\operatorname{map}(BV, Y)$ then $\pi_0 Z \cong \mathrm{Hom}_{\mathcal{K}}(H^* Y, H^* BV)$.*