Lecture 18: Proof of Lannes' theorem

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November 12, 2014

Let's recall some things from last time.

Definition 1. Let Y be so that H^*Y is finite type. Then (Z, ω) is a **model** for $\mathrm{map}(BV, Y)$ if

1. Z is a space of finite type.
2. $\omega : BV \times Z \to Y$ is a map so that $\widetilde{\omega}^* : H^*Z \to T_V H^*Y$ is an isomorphism.

Theorem 2. *If (Z, ω) is a model for $\mathrm{map}(BV, Y)$, then $(\mathbb{F}_p)_\infty Z \xrightarrow{\sim} \mathrm{map}(BV, (\mathbb{F}_p)_\infty Y)$.*

Lemma 3. *1. If Z is a space, then $H^0 Z = \mathbb{F}_p^{\pi_0 Z}$.*

2. $(T_V H^*Y)^0 \cong \mathbb{F}_p^{\mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*BV)}$.

In particular, if (Z, ω) is a model for $\mathrm{map}(BV, Y)$, then $\pi_0 Z \cong \mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*BV)$.

Recall (or use 1 of the lemma above to show) that if Z is of finite type, $y \in \pi_0 Z$, and $Z_y \subseteq Z$ is the associated path component, then $H^*Z_y = \mathbb{F}_p \otimes_{H^0 Z} H^*Z$, tensoring over the map

$$H^0 Z = \mathbb{F}_p^{\pi_0 Z} \xrightarrow{\epsilon_y} \mathbb{F}_p^{\{y\}} = \mathbb{F}_p.$$

If (Z, ω) is a model as above, and $\phi \in \mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*BV) = \pi_0 Z$, then

$$H^*Z_\phi = \mathbb{F}_p \otimes_{(T_V H^*Y)^0} T_V H^*Y =: T_V^\phi H^*Y.$$

Lemma 4. *Let $L \in \mathcal{K}$ have $L^0 \cong \mathbb{F}_p$. Then*

$$\mathrm{Hom}_{\mathcal{K}}(T_V^\phi H^*Y, L) = \mathrm{Hom}_{\mathcal{K}/\phi}(H^*Y, H^*BV \otimes L),$$

where the right-hand side means diagrams of the following form:

$$\begin{array}{ccc} H^*Y & \longrightarrow & H^*BV \otimes L \\ & \searrow^{\phi} & \downarrow \\ & & H^*BV = H^*BV \otimes L^0. \end{array}$$

This almost doesn't need proof, so I won't give one.

For example, take $L = H_*S^t$, $t \geq 1$. We get

$$\mathrm{Hom}_{\mathcal{K}}(T_V^\phi H^*Y, H^*S^t) \cong \mathrm{Hom}_{\mathcal{K}/\phi}(H^*Y, H^*BV \otimes H^*S^t). \quad (1)$$

This can be derived, but first, we need to establish some hypotheses.

Lemma 5. *Let $G_\bullet H^*Y \to H^*Y$ be the standard resolution of H^*Y in $\mathcal{K}$. Then $T_V^\phi G_\bullet H^*Y \to T_V^\phi H^*Y$.*

(What's 'resolution' mean? It has to be a simplicial object; at each level, it's the cohomology of a product of Eilenberg-Mac Lane spaces; and it has to be exact, meaning that its simplicial homotopy is just H^*Y in degree zero.)

Proof. $G : \mathcal{K} \to \mathcal{K}$ is UF, where U is the free unstable algebra functor and F is the forgetful functor to graded vector spaces. So this is of the form $U(\bigoplus F(n_\alpha))$, and

$$T_V U\left(\bigoplus F(n_\alpha)\right) = UT_V\left(\bigoplus F(n_\alpha)\right) = U\left(\bigoplus F(m_\beta)\right).$$

We need only check $\pi_* T_V G_\bullet H^*Y \cong T_V H^*Y$, which follows from the above because $\pi_* G_\bullet H^*Y \cong H^*Y$ and T_V is exact. To get the result for T_V^ϕ, tensor down, which is just picking out a summand and so is exact. □

Now apply this and (1) to get

$$\mathrm{Ext}^s_{\mathcal{K}}(T^\phi_V H^*Y, H^*S^t) \cong \mathrm{Ext}^s_{\mathcal{K}/\phi}(H^*Y, H^*BV \otimes H^*S^t) \cong R^s\,\mathrm{Der}_{\mathcal{K}}(H^*Y, \Sigma^t BV)_\phi.$$

Proof of Theorem 2. Let A, B, C be spaces. We claim there's a map $A \times \mathbb{F}_p B \to \mathbb{F}_p(A \times B)$: this is just $A \times \mathbb{F}_p B \to \mathbb{F}_p A \times \mathbb{F}_p B$, composed with the 'bilinear' map $\mathbb{F}_p A \times \mathbb{F}_p B \to \mathbb{F}_p(A \times B)$, which is morally $\mathbb{F}_p A \otimes \mathbb{F}_p B$.

Thus, given a map $A \times B \to C$, we get

$$A \times \mathbb{F}_p B \to \mathbb{F}_p(A \times B) \to \mathbb{F}_p C,$$

which has an adjoint

$$\mathbb{F}_p B \to \mathrm{map}(A, \mathbb{F}_p C).$$

That was all pretty formal' now take a model $\omega : BV times Z \to Y$ so that $\widetilde{\omega}^* : H^*Z \cong T_V H^*Y$. So we get a map of cosimplicial spaces

$$(\mathbb{F}_p)^\bullet Z \to \mathrm{map}(BV, \mathbb{F}_p^\bullet Y).$$

Let $\phi \in \pi_0 Z = \mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*BV)$ be a basepoint. This gives a based morphism of cosimplicial spaces. Take the component associated to ϕ. We have a map

$$\pi^s \pi_t (\mathbb{F}_p)^\bullet Z_\phi \to \pi^s \pi_t\,\mathrm{map}(BV, \mathbb{F}_p^\bullet Y)_\phi.$$

We need this to be an isomorphism for $t - s \geq 0$. The Bousfield-Kan spectral sequence will then give the result.

The key point here is just that T_V is exact and commutes with tensor products. We have isomorphisms

$$\begin{aligned}
\pi^s \pi_t \mathbb{F}_p^\bullet \mathbb{Z}_\phi &\cong \mathrm{Ext}^s_{\mathcal{K}}(H^*Z_\phi, H^*S^t) \\
&\cong \mathrm{Ext}^s_{\mathcal{K}}(T^\phi_V H^*Y, H^*S^t) \\
&\cong \mathrm{Ext}^s_{\mathcal{K}/\phi}(H^*Y, H^*BV \otimes H^*S^t) \\
&\cong \pi^s \pi_t\,\mathrm{map}(BV, (\mathbb{F}_p^\bullet) Y)_\phi.
\end{aligned}$$

The first and last lines come from the fact that, if $W = \mathbb{F}_p W_0$, then

$$\pi_t\,\mathrm{map}(Z, W)_\phi \cong \mathrm{Hom}_{\mathcal{K}/\phi}(H^*W, H^*W \otimes S^t).$$

This concludes the proof. □

Let's rewind. This all starts with Sullivan's fixed point conjecture (now theorem). Let X be a finite CW-complex and G a finite p-group (usually, the cyclic group of order p).

Theorem 6 (Sullivan conjecture)**.**

$$(\mathbb{F}_p)_\infty X^G \overset{\simeq}{\to} ((\mathbb{F}_p)_\infty X)^{\mathrm{h}G}.$$

This is nontrivial because the ordinary fixed points aren't homotopy invariant, but the right-hand side is; so under this finiteness hypothesis, the p-completion functor somehow creates homotopy-invariance.

We've already seen one case. If $G = V$ an $\mathbb{F}_p$-vector space and it acts trivially, then the statement is

$$(\mathbb{F}_p)_\infty X \cong \mathrm{map}(BV, (\mathbb{F}_p)_\infty X).$$

This is a pretty good theorem; we'll give Lannes' proof, because it makes it look easy, while Carlsson's makes it look hard. We'll end today with some generalities.

Recall that if G is a finite group, and X is a G-space. Then we have a map

$$X^G = \mathrm{map}_G(*, X) \to \mathrm{map}_G(EG, X) =: X^{\mathrm{h}G}.$$

We can think of EG as a cofibrant replacement for the point in the category of G-spaces. Filtering EG by skeleta gives a spectral sequence

$$H^s(G, \pi_t X) \Rightarrow \pi_{t-s} X^{\mathrm{h}G}.$$

If you think the above theorem is easy, try proving it, even in a very simple case, using this spectral sequence.

That was fixed points; now let's do orbits. We have

$$X_{\mathrm{h}G} = EG \times_G X \to * \times_G X = X/G.$$

Here $X \times_G Y$ is not a pullback, but the Borel construction

$$X \times_G Y = X \times Y/(gx, y) \sim (x, gy).$$

There is a fibration

$$X \to EG \times_G X \to BG,$$

and thus a Serre spectral sequence

$$H^s(G, H^t X) \Rightarrow H^{s+t} X_{\mathrm{h}G}.$$

Remark 7. The map from homotopy orbits to orbits is an equivalence when X is a free G-space, because this means that X is a cofibrant G-space, and so the ordinary orbits are already derived. The map from fixed points to homotopy fixed points is an equivalence when X is a fibrant G-space, which is a weirder condition.

Example 8. Let $G = \mathbb{Z}/p = C_p$. This has the nice property that it only has two subgroups. Thus, $X - X^G$ is a free C_p-space. Suppose that $X^G \subseteq X$ has a G-invariant NDR neighborhood U (it has a deformation retraction onto X^G). Then $X = U \cup V$ where $V = X - X^G$, and there is a pushout diagram

$$\begin{array}{ccccccccc}
(U \cap V)/G & \xleftarrow{\sim} & EG \times_G (U \cap V) & \longrightarrow & EG \times_G U & \xrightarrow{\sim} & EG \times_G X^G & = & BG \times X^G \\
\downarrow & & \downarrow & & \downarrow & & & & \\
V/G & \xleftarrow{\sim} & EG \times_G V & \longrightarrow & EG \times_G X & = & X_{\mathrm{h}G}. & &
\end{array}$$

So the actual fixed points show up in a pushout diagram involving the homotopy orbits.