Lecture 19: Orbits and homotopy orbits

Paul VanKoughnett

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We're discussing the Sullivan conjecture, which says that if π is a p-group, and X a finite CW-complex with a π-action, then
$$(\mathbb{F}_p)_\infty(X^\pi) \simeq ((\mathbb{F}_p)_\infty X)^{\mathrm{h}\pi}.$$
The crucial case is $\pi = \mathbb{Z}/p = C_p$. We could prove this right now, but it wouldn't be very enlightening. Instead, we'll work some examples.

Recall that if G acts on X, we have maps
$$X^G \to \mathrm{map}_G(EG, X) = X^{\mathrm{h}G}$$
and
$$X_{/G} \leftarrow EG \times_G X = X_{\mathrm{h}G}.$$
We also have a fiber sequence
$$X \to EG \times_G X \to BG.$$
Last itme, we noted that if $V = X - X^G$ was a free G-space, and $X^G \subseteq U$ a G-NDR pair, then
$$\begin{CD} U \cap V @>>> U \\ @VVV @VVV \\ V @>>> X \end{CD}$$
was a G-homotopy pushout. Taking homotopy G-orbits then gives a pushout
$$\begin{CD} (U \cap V)/G @>>> BG \times X^G \\ @VVV @VVV \\ EG \times_G V @>>> EG \times_G X. \end{CD}$$
Watch out: although $EG \times_G X$ is homotopy invariant, this square usually isn't.

For example, say $G = \mathbb{Z}$, generated by τ, and $X = \mathbb{R}$ with the G-action $\tau(x) = x + 1$. This has a non-equivariant homotopy equivalence to $Y = *$ with the trivial G-action. In X, we have $X^G = \emptyset$, which we might as well take to be U, and we get the square
$$\begin{CD} \emptyset @>>> \emptyset \\ @VVV @VVV \\ \mathbb{R}/\mathbb{Z} = EG \times_G \mathbb{R} @= EG \times_G \mathbb{R}. \end{CD}$$
For Y, we have $Y^G = Y$ and we instead take $V = \emptyset$. The square is
$$\begin{CD} \emptyset @>>> EG \times_G * = S^1 \\ @VVV @VVV \\ \emptyset @>>> EG \times_G *. \end{CD}$$

We get two models for BG, which are equivalent but very nontrivially so: it's an equivalence of the form

$$S^1 \simeq \mathbb{R}/G \longleftarrow EG \times_G \mathbb{R} \xrightarrow{\simeq} EG \times_G * =\!=\!= B\mathbb{Z} \cong S^1$$

$$[y] \longleftarrow\!\!\shortmid (x,y) \longmapsto (x,*) \longmapsto [x].$$

Example 1. Let $G = C_2 = \{1, \tau\}$. Let $D^{i+1} \subseteq \mathbb{R}^{i+1}$ be the unit disk, so $\partial D^{i+1} = S^i$. We have

$$S^{i+j+1} = \partial D^{i+j+2} = \partial(D^{i+1} \times D^{j+1}) = S^i \times D^{j+1} \cup D^{i+1} \times S^j.$$

The intersection is $S^i \times D^{j+1} \cap D^{i+1} \times S^j = S^i \times S^j$. We get a homotopy pushout diagram

$$\begin{array}{ccc} S^i \times S^j & \longrightarrow & S^i \times D^{j+1} \\ \downarrow & & \downarrow \\ D^{i+1} \times S^j & \longrightarrow & S^{i+j+1}. \end{array} \tag{1}$$

If you're an old-fashioned homotopy theorist, you just proved that S^{i+j+1} is the join of S^i and S^j.

Now let's make this C_2-equivariant. Let C_2 act on $D^{i+1} \times D^{j+1}$ by $\tau(x,y) = (-x,y)$. Thus, $\tau|_{S^i}$ is the antipodal map, which is fixed-point free and has degree $(-1)^{i+1}$; $\tau|_{S^j}$ is the identity map, with degree 1. Using the Mayer-Vietoris sequence, we can show that τ has degree $(-1)^{i+1}$ on S^{i+j+1}. Finally, (1) is a C_2-equivariant diagram.

Remark 2. The Serre spectral sequence

$$H^*(C_2, H^* S^{i+j+1}) \Rightarrow H^*(EC_2 \times_{C_2} S^{i+j+1})$$

only depends on i mod 2. If $i \equiv 1 \pmod 2$, then $(-1)^{i+1}$ is odd, and the E_2 page of the spectral sequence is a copy of $H^*(BC_2, \mathbb{Z}) \cong \mathbb{Z}[x_2]/(2x)$ on each of the rows $q = 0$ and $q = i+j+1$. S^{i+j+1} has a fixed point, so there's a section to $EC_2 \times_{C_2} S^{i+j+1} \to BC_2$, which means that there can't be any differentials, and the spectral sequence collapses here. One might guess that

$$EC_2 \times_{C_2} S^{i+j+1} \simeq BC_2 \times S^{i+j+1}.$$

In fact, this is false, but one needs the diagram (1) to do it. Indeed, the homotopy orbits of (1) are

$$\begin{array}{ccc} (S^i \times S^j)_{\mathrm{h}C_2} & \longrightarrow & (D^{i+1} \times S^j)_{\mathrm{h}C_2} \\ \downarrow & & \downarrow \\ (S^i \times D^{j+1})_{\mathrm{h}C_2} & \longrightarrow & (S^{i+j+1})_{\mathrm{h}C_2}. \end{array}$$

The action on $S^i \times S^j$ and $S^i \times D^{j+1}$ is free, so the homotopy orbits are the orbits. On the other hand, $D^{i+1} \times S^j$ is C_2-homotopy equivalent to S^j with the trivial action. Thus, the diagram is

$$\begin{array}{ccccccc} \mathbb{R}P^i \times S^j & \xleftarrow{\simeq} & (S^i \times S^j)_{\mathrm{h}C_2} & \longrightarrow & (D^{i+1} \times S^j)_{\mathrm{h}C_2} & \xrightarrow{\simeq} & \mathbb{R}P^\infty \times S^j \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{R}P^i \times D^{j+1} & \xleftarrow{\simeq} & (S^i \times D^{j+1})_{\mathrm{h}C_2} & \longrightarrow & (S^{i+j+1})_{\mathrm{h}C_2}. & & \end{array}$$

Notice that the left-hand column is all finite complexes, which are ignored by the T-functor; the upper right corner is the fixed points, and the lower right corner is the homotopy orbits.

Lemma 3. *The top map $\mathbb{R}P^i \times S^j \to \mathbb{R}P^\infty \times S^j$ is homotopic to $\eta \times 1$, where $\eta : \mathbb{R}P^i \to \mathbb{R}P^\infty$ is the inclusion map.*

This isn't hard to check: the map is clearly the identity on the S^j factor, and then you have to sit down and figure out what it does to the free part, just as we did with $B\mathbb{Z} = S^1$ earlier.

Now, the pushout square on homotopy orbits gives a Mayer-Vietoris sequence for $H^*(EC_2 \times_{C_2} S^{i+j+1})$:

$$\cdots \to H^*(S^{i+j+1}_{\mathrm{h}C_2}) \to H^*(\mathbb{R}P^i) \times H^*(\mathbb{R}P^\infty \times S^j) \to H^*(\mathbb{R}P^i \times S^j) \to \cdots .$$

If i is odd, then the orientation class $\mathbb{Z} \in H^{i+j}(\mathbb{R}P^i \times S^j)$ is not in the image of $H^*(\mathbb{R}P^i) \times H^*(\mathbb{R}P^\infty \times S^j)$, so it gives something in $H^{i+j+1}(S^{i+j+1}_{\mathrm{h}C_2})$, which we saw in the spectral sequence. If i is even, then the Mayer-Vietoris splits up into short exact sequences.

For the Sullivan conjecture for $\pi = C_p = \mathbb{Z}/p$, we're going to work with the same picture:

$$\begin{array}{ccc} (U \cap V)/C_p & & EC_p \times_{C_p} X^{C_p} = BC_p \times X^{C_p} \\ & & \big\downarrow \\ V/C_p & \longrightarrow & EC_p \times_{C_p} X. \end{array}$$

Here are the crucial steps: we'll use Lannes' comparison theorem calculate $\mathrm{map}(BC_p, EC_p \times_{C_p} X)$ and relate it to $\mathrm{map}(BC_p, BC_p \times X^{C_p})$. Since X^{C_p} is a finite complex, we'll get

$$\mathrm{map}(BC_p, BC_p \times X^{C_p}) = \mathrm{map}(BC_p, BC_p) \times \mathrm{map}(BC_p, X^{C_p}) \simeq X^{C_p},$$

at least up to p-completion. Then the previous arguments will finish the result.