Lecture 2: Admissible sequences

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Last time, we introduced the Steenrod squares. These were natural group homomorphisms

$$\mathrm{Sq}^i : H^n X \to H^{n+i} X$$

on the mod 2 cohomology of a space X, subject to some axioms. First, the squares are **natural**, meaning that diagrams of the form

$$\begin{array}{ccc} H^n Y & \xrightarrow{\mathrm{Sq}^i} & H^{n+i} Y \\ \downarrow f^* & & \downarrow f^* \\ H^n X & \xrightarrow{\mathrm{Sq}^i} & H^{n+i} Y \end{array}$$

commute, where $f : X \to Y$ is a map of spaces.

For example, suppose that X and Y are **finite type**, meaning that $\dim H^n X < \infty$ for all n. Then there's a Künneth isomorphism

$$H^* X \otimes H^* Y \to H^*(X \times Y).$$

We'll write $x \times y$ for the image of $x \otimes y$ under this map. The projection map $p_1 : X \times Y \to X$ induces $p_1^* : H^* X \to H^*(X \times Y)$ sending $a \mapsto a \times 1$. By naturality, $\mathrm{Sq}^i(a) \mapsto \mathrm{Sq}^i(a \times 1)$. Thus, in $H^*(X \times Y)$,

$$\mathrm{Sq}^i(x \times y) = \mathrm{Sq}^i((x \times 1) \cup (1 \times y)) = \sum_{j+k=i} \mathrm{Sq}^j(x \times 1) \cup \mathrm{Sq}^k(1 \times y)$$

by the Cartan formula, so naturality has given us

$$\mathrm{Sq}^i(x \times y) = \sum_{j+k=i} (\mathrm{Sq}^j x) \times (\mathrm{Sq}^k y).$$

Another property of the squares is the Adem relations:

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{2t \le i} \binom{j-t-1}{i-2t} \mathrm{Sq}^{i+j-t} \mathrm{Sq}^t .$$

Example 1.

$$\begin{aligned} \mathrm{Sq}^4 \mathrm{Sq}^8 &= \sum_{t=0}^{2} \binom{7-t}{4-2t} \mathrm{Sq}^{12-t} \mathrm{Sq}^t \\ &= \binom{7}{4} \mathrm{Sq}^{12} + \binom{6}{2} \mathrm{Sq}^{11} \mathrm{Sq}^1 + \binom{5}{3} \mathrm{Sq}^{10} \mathrm{Sq}^2 \\ &= \mathrm{Sq}^{12} + \mathrm{Sq}^{11} \mathrm{Sq}^1 . \end{aligned}$$

We note that $4 < 8$, but the exponents on the right-hand side decrease in each monomial.

Definition 2. $\mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_s}$ is **admissible** if $i_k \ge 2 i_{k+1}$ for each $1 \le k < s$.

For example, $\mathrm{Sq}^6 \mathrm{Sq}^3 \mathrm{Sq}^1$ is admissible.

By the Adem relations, if $\mathrm{Sq}^i \mathrm{Sq}^j$ is not admissible, we can rewrite it as a sum of admissible terms. More explicitly, if $i < 2j$, then the Adem relations rewrite $\mathrm{Sq}^i \mathrm{Sq}^j$ as a sum of terms of the form $\mathrm{Sq}^{i+j-t} \mathrm{Sq}^t$ with $t \leq i/2$. Thus, we have

$$2t \leq i = i/2 + i/2 < i/2 + j \leq i + j - t.$$

More generally,

Proposition 3. *If* $\mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_s}$ *is not admissible, it can be rewritten as a sum of admissibles.*

The proof will be given below.

Definition 4. The **Steenrod algebra** $\mathcal{A}$ is the graded (associative, noncommutative) tensor algebra over $\mathbb{F}_2$ on the Sq^i, where $|\mathrm{Sq}^i| = i$, mod the Adem relations.

Example 5. Here's a basis for $\mathcal{A}$ in low degrees:

degree	basis		
0	1		
1	Sq^1		
2	Sq^2,	$(\mathrm{Sq}^1 \mathrm{Sq}^1 = 0)$	
3	$\mathrm{Sq}^3 = \mathrm{Sq}^1 \mathrm{Sq}^2$,	$\mathrm{Sq}^2 \mathrm{Sq}^1$	
4	Sq^4,	$\mathrm{Sq}^3 \mathrm{Sq}^1 = \mathrm{Sq}^2 \mathrm{Sq}^2$	
5	Sq^5,	$\mathrm{Sq}^4 \mathrm{Sq}^1$	
6	Sq^6,	$\mathrm{Sq}^5 \mathrm{Sq}^1$,	$\mathrm{Sq}^4 \mathrm{Sq}^2$

We can now rewrite Proposition 3.

Proposition 6. *The admissible monomials form a basis for the Steenrod algebra.*

Proof. First, let's show that they span. If $I = (i_1, \ldots, i_s)$, write $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_s}$. Order the symbols I lexicographically, so that $(7, 0) > (6, 1) > (5, 2)$. By the Adem relations, if Sq^I is not admissible, we can write $\mathrm{Sq}^I = \sum_J a_J \mathrm{Sq}^J$, where all $J > I$. This process preserves degrees, and there are only finitely many monomials in each degree, so after doing this a finite number of times, we must get to a linear combination of admissibles.

Now let's show that they're linearly independent. Look at the space $X = (\mathbb{R}P^\infty)^\times n$. Then $H^*X = \mathbb{F}_2[x_1, \ldots, x_n]$. Now write $y = x_1 \cdots x_n$.

Exercise 7. The elements $\mathrm{Sq}^I(x_1 \cdots x_n) \in H^*X$, with I admissible and $e(I) := i_1 - i_2 - \cdots - i_s \leq n$, is linearly independent.

Now the Steenrod algebra acts on the cohomology of any space, so since these things are linearly independent in some space, they're linearly independent in $\mathcal{A}$. □

Another fact about the Steenrod operations was the **unstable condition**: if $i > |x|$, then $\mathrm{Sq}^i x = 0$. In particular, $\mathrm{Sq}^i \mathrm{Sq}^j(x) = 0$ if $i > |\mathrm{Sq}^j x| = |x| + j$, i. e. if $i - j > |x|$.

Definition 8. The **excess** of an sequence I is

$$e(I) = i_1 - i_2 - \cdots - i_s.$$

If $e(I) > |x|$, then $\mathrm{Sq}^I(x) = 0$, by the same logic.

If I is admissible, it's convenient to write

$$e(I) = (i_1 - 2i_2) + \cdots + (i_{s-1} - 2i_s),$$

a sum of positive numbers measuring how admissible I is.

Let's define

$$\mathsf{Mod}_{\mathcal{A}} = \text{ the category of (graded) left } \mathcal{A}\text{-modules},$$

and

$$\mathcal{U} \subseteq \mathsf{Mod}_{\mathcal{A}} = \text{ the full subcategory of } \textbf{unstable modules},$$

those modules M such that $\mathrm{Sq}^i(x) = 0$ if $|x| < i$ for $x \in M$. Finally,

$$\mathcal{K} = \text{ the category of } \textbf{unstable algebras} \text{ over } \mathcal{A}.$$

An unstable algebra is an unstable module with a graded algebra structure, such that $\mathrm{Sq}^{|x|}(x) = x^2$, where the morphisms preserve that $\mathcal{A}$-module and algebra structures. The functor H^* has image in $\mathcal{K}$.

There are adjunctions

$$\mathcal{K} \overset{U}{\rightleftarrows} \mathcal{U} \overset{\Omega^\infty}{\rightleftarrows} \mathsf{Mod}_{\mathcal{A}}.$$

The right adjoints are the obvious forgetful functors; U stands for 'universal enveloping algebra'; Ω^∞ refers to the infinite loop space functor from topology. What does this adjunction thing mean? Well, if $M \in \mathcal{U}$ and $N \in \mathsf{Mod}_{\mathcal{A}}$, then

$$\mathrm{Hom}_{\mathcal{A}}(N, M) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega^\infty N, M).$$

Thus, $\Omega^\infty N$ should be the free unstable $\mathcal{A}$-module with a map from N, which should be

$$\Omega^\infty N = N / \{\mathrm{Sq}^i x : i > |x|\}.$$

Exercise 9. Show that $\{\mathrm{Sq}^i x : i > |x|\}$ is, in fact, a sub-$\mathcal{A}$-module of $\mathcal{N}$, using the Adem relations.

Now let's construct U. Let V be a graded vector space over $\mathbb{F}_2$. The **symmetric algebra** over V is

$$S(V) = \bigoplus_{i \geq 0} V^{\otimes n} / \Sigma_n.$$

(Note we're constructing this to be graded commutative, not ordinarily commutative.) If $x_1, \ldots, x_n$ is a basis, then $S(V) \cong \mathbb{F}_2[x_1, \ldots, x_n]$. The symmetric algebra has the following universal property: any map $V \to B$ to a graded commutative algebra B extends uniquely to an algebra map $S(V) \to B$.

In particular, if $M \in \mathcal{U}$, then we define $U(M) = S(M)/(\mathrm{Sq}^{|x|} + x^2)$.

Lemma 10. *Thus constructed, $U(M) \in \mathcal{K}$.*

Exercise 11. Prove this. You need to show that the ideal $(\mathrm{Sq}^{|x|} + x^2)$ is closed under the Steenrod action.

Theorem 12 (Serre's thesis)**.** *If $K(\mathbb{F}_2, n)$ is an Eilenberg-Mac Lane space, then the natural map*

$$U\Omega^\infty \Sigma^n \mathcal{A} \to H^* K(\mathbb{F}_2, n),$$

induced from the map of $\mathcal{A}$-modules

$$\Sigma^n \mathcal{A} \to H^* K(\mathbb{F}_2, n)$$

that sends $a \mapsto a(\iota_n)$, is an isomorphism.

(Notation: the Σ^n is a shift up by n degrees, and $\iota_n \in H^n K(\mathbb{F}_2, n)$ is the fundamental class.)