Lecture 20: Orbits and homotopy orbits

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Recall that we have C_2 acting on S^{i+j+1} by the antipodal map on the first $(i+1)$ coordinates. We wrote the homotopy orbits as a pushout square

$$\begin{array}{ccc} \mathbb{R}P^j \times S^i & \longrightarrow & \mathbb{R}P^\infty \times S^i \\ {\scriptstyle p_j}\downarrow & & \ulcorner\downarrow \\ \mathbb{R}P^j & \longrightarrow & EC_2 \times_{C_2} S^{i+j+1}. \end{array}$$

In mod 2 cohomology, there's a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(EC_2 \times_{C_2} S^{i+j+1}) & \longrightarrow & H^*\mathbb{R}P^j \times H^*(\mathbb{R}P^\infty \times S^i) & \longrightarrow & H^*\mathbb{R}P^j \times S^i & \longrightarrow & 0 \\ & & & & \downarrow\cong & & \downarrow\cong & & \\ & & & & \mathbb{F}_2[x]/(x^{j+1}) \times \mathbb{F}_2[x] \otimes \Lambda[y] & \longrightarrow & \mathbb{F}_2[x]/(x^{j+1}) \otimes \Lambda(y) & \longrightarrow & 0 \end{array}$$

We need to compute the kernel of the bottom map. We have $(x, x)^n = (x^n, x^n) \mapsto 0$, and $(0, x^n y) \mapsto x^n y$, which is 0 for $n \geq j + 1$. Let $\mu = (x, x)$, and $z = (0, x^{j+1}y)$. We get

$$H^*(EC_2 \times_{C_2} S^{i+j+1}) \cong \mathbb{F}_2[\mu] \otimes \Lambda(z).$$

The Steenrod operations act by $\mathrm{Sq}^i \mu^j = \binom{j}{i}\mu^{i+j}$, and $\mathrm{Sq}^i(z) = \mathrm{Sq}^i(x^{j+1}y) = \binom{j+1}{i}x^{i+j+1}y$. So, although this cohomology is the same as that of $H^*(\mathbb{R}P^\infty \times S^{i+j+1})$ as a ring, it's generally not the same as an unstable algebra.

The Sullivan conjecture, in its most basic form, is

$$(\mathbb{F}_p)_\infty(X^{\mathbb{Z}/p}) \simeq ((\mathbb{F}_p)_\infty X)^{\mathrm{h}\mathbb{Z}/p}.$$

To prove this, we're going to calculate $\mathrm{map}(B\mathbb{Z}/p, E\mathbb{Z}/p \times_{\mathbb{Z}/p} (\mathbb{F}_p)_\infty X)$, and see what we get.

There's a fiber sequence

$$X \to E\mathbb{Z}/p \times_{\mathbb{Z}/p} X = X_{\mathrm{h}\mathbb{Z}/p} \to B\mathbb{Z}/p$$

so we get a map

$$\mathrm{map}(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p}) \to \mathrm{map}(B\mathbb{Z}/p, B\mathbb{Z}/p) \simeq \coprod_{\phi:\mathbb{Z}/p\to\mathbb{Z}/p} B\mathbb{Z}/p.$$

To get the expression on the right, note that ϕ is the same as a representation of $\mathbb{Z}/p$ in itself, and its centralizer is all of $\mathbb{Z}/p$. Pulling this decomposition back, we can break $\mathrm{map}(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p})$ up as a coproduct over ϕ of components which we'll call $\mathrm{map}_\phi(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p})$. These components are not created equal. If $\phi = 0$, we have a fiber sequence

$$\mathrm{map}(B\mathbb{Z}/p, X) \to \mathrm{map}_0(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p}) \to B\mathbb{Z}/p = \mathrm{map}_0(B\mathbb{Z}/p, B\mathbb{Z}/p),$$

where $\mathrm{map}(B\mathbb{Z}/p, X)$ is the fiber over $\mathrm{map}(B\mathbb{Z}/p, *) = *$. If X is finite, then $\mathrm{map}(B\mathbb{Z}/p, X) \simeq X$, and we might guess that $\mathrm{map}_0(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p}) \simeq X_{\mathrm{h}\mathbb{Z}/p}$.

If $\phi \neq 0$, the basepoint no longer lives in the ϕ component of $\mathrm{map}(B\mathbb{Z}/p, B\mathbb{Z}/p)$. Instead, we pick a basepoint $\{\phi\}$, and let $F_\phi \subseteq \mathrm{map}_\phi(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p})$ be the fiber. This is the space of maps

$$\begin{array}{ccc} B\mathbb{Z}/p & \longrightarrow & E\mathbb{Z}/p \times_{\mathbb{Z}/p} X \\ & \overset{\phi}{\searrow} & \downarrow \\ B\mathbb{Z}/p. & & \end{array}$$

For simplicity, let's say $\pi_1 X = 0$. By covering space theory,

$$F_\phi = \mathrm{map}_{\mathbb{Z}/p}(E\mathbb{Z}/p, E\mathbb{Z}/p \times X) \simeq X^{\mathrm{h}\mathbb{Z}/p}.$$

This is very different than the case $\phi = 0$. It is independent of ϕ, but the equivalence isn't – it's a matter of choosing a basepoint of $B\mathbb{Z}/p$, which changes the mapping space by conjugation.

We've gotten the homotopy fixed points $X^{\mathrm{h}\mathbb{Z}/p}$ as fibers living inside $\mathrm{map}(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p})$. Now we'll use Lannes' theorem to calculate $\mathrm{map}(B\mathbb{Z}/p, ((\mathbb{F}_p)_\infty X)_{\mathrm{h}\mathbb{Z}/p})$. This requires us to produce a model

$$\omega : B\mathbb{Z}/p \times Z \to X_{\mathrm{h}\mathbb{Z}/p}$$

so that the adjoint map

$$\widetilde{\omega}_* : H^*Z \leftarrow TH^*X_{\mathrm{h}\mathbb{Z}/p}$$

is an isomorphism.

Assume that X is finite. We'll let

$$Z = X_{\mathrm{h}\mathbb{Z}/p} \sqcup \coprod_{\phi \neq 0} B\mathbb{Z}/p \times X^{\mathbb{Z}/p}.$$

The map ω is defined on each component as follows. On the zero component, we just take the projection

$$B\mathbb{Z}/p \times X_{\mathrm{h}\mathbb{Z}/p} \to X_{\mathrm{h}\mathbb{Z}/p}.$$

The adjoint map $X_{\mathrm{h}\mathbb{Z}/p} \to \mathrm{map}(B\mathbb{Z}/p, X_{\mathrm{h}\mathbb{Z}/p})$ sends x to the constant map to x. On a component corresponding to $\phi \neq 0$, ω is

$$B\mathbb{Z}/p \times (B\mathbb{Z}/p \times X^{\mathbb{Z}/p}) \to B\mathbb{Z}/p \times X^{\mathbb{Z}/p} \subseteq E\mathbb{Z}/p \times_{\mathbb{Z}/p} X,$$

defined by $(a, b, x) \mapsto (\phi(a) + b, x)$.

We need to show that $\widetilde{\omega}^*$ is an isomorphism. This requires us to calculate $TH^*X_{\mathrm{h}\mathbb{Z}/p}$. Recall that $H^*B\mathbb{Z}/p \cong \widetilde{H}^*B\mathbb{Z}/p \oplus \mathbb{F}_p$, so

$$\mathrm{Hom}_{\mathcal{U}}(TK, L) \cong \mathrm{Hom}_{\mathcal{U}}(K, \widetilde{H}^*B\mathbb{Z}/p \otimes L) \times \mathrm{Hom}_{\mathcal{U}}(K, L) \cong \mathrm{Hom}_{\mathcal{U}}(\overline{T}K \times K, L),$$

where we've defined $\overline{T}$ to be left adjoint to $\widetilde{H}^*B\mathbb{Z}/p \otimes \cdot$. If K is finite, we know already that $TK = K$, so $\overline{T}K = 0$.

Let $U \supseteq X^{\mathbb{Z}/p}$ be a $\mathbb{Z}/p$-NDR. (We'd have to check that these exist.) There's a homotopy pushout

$$\begin{array}{ccc} (U - X^{\mathbb{Z}/p})/(\mathbb{Z}/p) & \longrightarrow & B\mathbb{Z}/p \times X^{\mathbb{Z}/p} \\ \downarrow & & \ulcorner \downarrow \\ (X - X^{\mathbb{Z}/p})/(\mathbb{Z}/p) & \longrightarrow & X_{\mathrm{h}\mathbb{Z}/p}. \end{array}$$

The spaces on the left-hand side are finite, so $\overline{T}$ kills them; $\overline{T}$ is also exact, so we get

Proposition 1.

$$\overline{T}H^*X_{\mathrm{h}\mathbb{Z}/p} \cong \overline{T}H^*(B\mathbb{Z}/p \times X^{\mathbb{Z}/p}).$$

In other words,

$$TH^*X_{\mathrm{h}\mathbb{Z}/p} = H^*X_{\mathrm{h}\mathbb{Z}/p} \oplus \overline{T}H^*(B\mathbb{Z}/p \times X^{\mathrm{h}\mathbb{Z}/p}).$$

Hopefully this is starting to look familiar.

We calculate that $T(H^*B\mathbb{Z}/p \otimes H^*X^{\mathbb{Z}/p}) = T(H^*B\mathbb{Z}/p) \otimes H^*X^{\mathbb{Z}/p}$, since $T(A \otimes B) = T(A) \otimes T(B)$ and $X^{\mathbb{Z}/p}$ is finite. And, of course, $T(H^*B\mathbb{Z}/p) \cong H^*B\mathbb{Z}/p \times \prod_{\phi \neq 0} H^*B\mathbb{Z}/p$, so $\overline{T}(H^*B\mathbb{Z}/p) \cong \prod_{\phi \neq 0} H^*B\mathbb{Z}/p$. We get

$$\overline{T}(H^*B\mathbb{Z}/p \otimes H^*X^{\mathbb{Z}/p}) \cong \prod_{\phi \neq 0} H^*B\mathbb{Z}/p \otimes H^*X^{\mathbb{Z}/p}.$$

At this point, we're basically done. We have to check that the adjoint maps are what we expect them to be, but that's not too hard. This is a testament to the usefulness of the exactness and monoidality of T.

By Lannes' theorem, if (Z, ω) is our model above,

$$(\mathbb{F}_p)_\infty Z \simeq \mathrm{map}(B\mathbb{Z}/p, (\mathbb{F}_p)_\infty X_{\mathrm{h}\mathbb{Z}/p}).$$

The left-hand side is

$$(\mathbb{F}_p)_\infty X_{\mathrm{h}\mathbb{Z}/p} \sqcup \coprod_{\phi \neq 0} B\mathbb{Z}/p \times (\mathbb{F}_p)_\infty X^{\mathbb{Z}/p}$$

($B\mathbb{Z}/p$ is already p-complete). The right-hand side is

$$\mathrm{map}(B\mathbb{Z}/p, ((\mathbb{F}_p)_\infty X)_{\mathrm{h}\mathbb{Z}/p})$$

by the 'nilpotent fiber lemma,' which will be proved momentarily. Now take the fiber at $1 \in \mathrm{map}(B\mathbb{Z}/p, B\mathbb{Z}/p)$ on both sides. We get

$$(\mathbb{F}_p)_\infty (X^{\mathbb{Z}/p}) \simeq \mathrm{map}_{\mathbb{Z}/p}(E\mathbb{Z}/p, (\mathbb{F}_p)_\infty X),$$

which is the Sullivan conjecture.

We still need to prove the nilpotent fiber lemma, which goes as follows.

Lemma 2. *Let $F \to X \to Y$ be a fibration sequence of connected spaces, and suppose $\pi_1 Y$ acts nilpotently on $\pi_* F$. Then*

$$(\mathbb{F}_p)_\infty F \to (\mathbb{F}_p)_\infty X \to (\mathbb{F}_p)_\infty Y$$

is still a fiber sequence.

Applying this to $X \to E\mathbb{Z}/p \times_{\mathbb{Z}/p} X \to B\mathbb{Z}/p$, we get $(\mathbb{F}_p)_\infty X_{\mathrm{h}\mathbb{Z}/p} \simeq ((\mathbb{F}_p)_\infty X)_{\mathrm{h}\mathbb{Z}/p}$.