Lecture 21: Nilpotent spaces and p-completion

Paul VanKoughnett

November 19, 2014

For the record, the Sullivan conjecture is true for an arbitrary finite p-group π:

$$(\mathbb{F}_p)_\infty(X^\pi) \simeq ((\mathbb{F}_p)_\infty X)^{\mathrm{h}\pi}.$$

The very first fact you learn about p-groups is that they have a nontrivial center. Let $x \in Z(\pi)$ be a central element, and let σ be the subgroup it generates. This is a normal subgroup! So we have a smaller p-group $\pi' = \pi/\sigma$; also, $X^\pi = (X^\sigma)^{\pi'}$, and likewise for homotopy orbits. So we get

$$(\mathbb{F}_p)_\infty(X^\pi) = (\mathbb{F}_p)_\infty((X^\sigma)^{\pi'}),$$

and by induction on the order of π, this is

$$((\mathbb{F}_p)_\infty(X^\sigma))^{\mathrm{h}\pi'} = (((\mathbb{F}_p)_\infty X)^{\mathrm{h}\sigma})^{\mathrm{h}\pi'} = ((\mathbb{F}_p)_\infty X)^{\mathrm{h}\pi}.$$

So there's really nothing in this more complicated case that requires anything.

In particular, if π acts trivially, $(\mathbb{F}_p)_\infty X \simeq \mathrm{map}(B\pi, (\mathbb{F}_p)_\infty X)$.

Let's change tacks and talk some more about Bousfield-Kan p-completion.

Definition 1. Let G be a group. G is **nilpotent** if one of the following equivalent conditions holds:

1. The lower central series terminates;
2. G has a finite filtration
$$\{e\} \lhd G_n \lhd \cdots \lhd G_1 \lhd G$$
(each G_i normal in G_{i-1}) with G_{i-1}/G_i abelian.

Any solvable group is nilpotent, but, for example, the free group on 2 generators, $\pi_1(S^1 \vee S^1)$, is not: its lower central series has associated graded the free Lie algebra on 2 generators, which is pretty big.

Definition 2. Let G be a group and M a G-module (an abelian group with an action of G). Then M is **nilpotent** if it has a finite filtration by G-modules

$$0 \subseteq M_n \subseteq \cdots \subseteq M_1 \subseteq M$$

where G acts trivially on M_i/M_{i+1}.

We can even generalize this to where M is nonabelian; then a nilpotent group is a group that's nilpotent for its own conjugation action.

Definition 3. A connected space is **nilpotent** if $\pi_1 X$ is nilpotent and acts nilpotently on $\pi_n X$, $n > 1$.

For example, any simply connected space is nilpotent. On the other hand, even if $\pi_1 X$ is abelian, X might not be nilpotent.

Example 4. $S^1 \vee S^2$ is not a nilpotent space. Its universal cover is $\mathbb{R}$ with a copy of S^2 glued to each integer. Thus $\pi_2(S^1 \vee S^2) = \mathbb{Z}[\pi_1]$, where $\pi_1 \cong \mathbb{Z}$, say generated by τ. If $f : \mathbb{Z}[\pi_1] \twoheadrightarrow M$ is a surjection onto a trivial π_1-module, then $\ker f$ contains $1 - \tau^i$ for all i, so $\ker(f)$ contains the augmentation ideal I of $\mathbb{Z}[\pi_1]$. But this means that M is a quotient of the integers, which are $\mathbb{Z}[\pi_1]/I$. In fact, I is isomorphic to $\mathbb{Z}[\pi_1]$, via $x \mapsto x(1-\tau) \in I$. So we won't have the finite filtration we need.

Theorem 5. *Let X be a nilpotent space. Then $X \to (\mathbb{F}_p)_\infty X$ is the $H_*(\cdot;\mathbb{F}_p)$-localization of X, i. e. the terminal homology isomorphism out of X.*

In other words, if $X \to Y$ is a mod p homology isomorphism, then there's a unique map $Y \to (\mathbb{F}_p)_\infty X$ such that the composition is $X \to (\mathbb{F}_p)_\infty X$. The existence of such a localization (for any homology theory, not just ordinary homology) was proved by Bousfield in an important paper in *Topology*, 1975.

Example 6 (Bousfield). $(S^1 \vee S^n) \to (\mathbb{F}_p)_\infty(S^1 \vee S^n)$, where $n \geq 2$, is not an isomorphism on mod p homology. In fact, none of the maps

$$(\mathbb{F}_p)_\infty^k(S^1 \vee S^n) \to (\mathbb{F}_p)_\infty^{k+1}(S^1 \vee S^n)$$

are homology isomorphisms. Dwyer showed that you get a homology isomorphism if you do this transfinitely many times.

Proposition 7. *Let X be simply connected.*

1. *If $\pi_n X$ is finitely generated, then $\pi_n(\mathbb{F}_p)_\infty X = (\pi_n X)_p^\wedge$.*
2. *More generally, there is a split short exact sequence*

$$0 \to L_0\pi_n X \to \pi_n(\mathbb{F}_p)_\infty X \to L_1\pi_{n-1}X \to 0$$

where L_iA is the ith left derived functor of completion.

You may have heard that completion is left exact. This only happens under finite type hypotheses – in general, the functor L_0 is not actually completion. Let's digress to talk about this.

For an abelian group A, we have $A_p^\wedge = \lim(\mathbb{Z}/p^n \otimes A)$. In general, $\mathbb{Z}/p^n \otimes \cdot$ has left derived functors, and lim has right derived functors. So we have $L_1A = \lim \mathrm{Tor}_1(\mathbb{Z}/p^n, A)$, and L_0A fits into an exact sequence

$$0 \to \lim(\mathbb{Z}/p^n \otimes A) \to L_0A \to \overset{1}{\lim}\,\mathrm{Tor}_1(\mathbb{Z}/p^n, A) \to 0.$$

The higher derived functors are zero.

For example, let $\mathbb{Z}/p^\infty = \mathrm{colim}\,\mathbb{Z}/p^n \subseteq \mathbb{Q}/\mathbb{Z}$. One can show that $L_0\mathbb{Z}/p^\infty = 0$, and that $L_1\mathbb{Z}/p^\infty = \mathbb{Z}_p$. As a result, $(\mathbb{F}_p)_\infty K(\mathbb{Z}/p^\infty, n) = K(\mathbb{Z}_p, n+1)$. (These are good things to do in your chair when the lecture gets boring.)

Lemma 8 (Nilpotent fiber lemma)**.** *Let $p : E \to B$ be a fibration with E, B connected and $\pi_1E \to \pi_1B$ onto. If π_1B acts nilpotently on $H_*(F, \mathbb{F}_p)$, then the natural map*

$$(\mathbb{F}_p)_\infty F \to \mathrm{fiber}\{(\mathbb{F}_p)_\infty E \to (\mathbb{F}_p)_\infty B\}$$

is a weak equivalence. That is, $(\mathbb{F}_p)_\infty$ preserves this fibration.

We can use this to study the p-completion functor on nilpotent spaces, by working up the Postnikov tower.

Lemma 9. *Let A be an abelian group. Then*

$$K(A, n) \to (\mathbb{F}_p)_\infty K(A, n)$$

is a homology isomorphism, and

$$\pi_n(\mathbb{F}_p)_\infty K(A, n) = L_0A,$$
$$\pi_{n+1}(\mathbb{F}_p)_\infty K(A, n) = L_1A.$$

Proof. You can prove this by direct calculation using the Bousfield-Kan spectral sequence. Bousfield has a slicker way of doing it by defining a universal property for the right-hand side. □

If X is nilpotent, then we can write $X = \operatorname{holim} X_n$, where $\{X_n\}$ is the 'refined' Postnikov tower, a tower of fibrations

$$\cdots \twoheadrightarrow X_2 \twoheadrightarrow X_1 \twoheadrightarrow * = X_0$$

where each $X_n \to X_{n-1}$ has fiber $K(A, s_n)$ and $\pi_1 X_{n-1}$ acts trivially on $H_* K(A, s_n)$. (Note that $s \neq n$ in general: we need to filter each of the homotopy groups of X into pieces with a trivial π_1-action.) Moreover, s_n is increasing, and hits each integer k only finitely many times.

By induction and the nilpotent fiber lemma, we get that

$$X_n \to (\mathbb{F}_p)_\infty X_n$$

is a mod p homology isomorphism. After showing that p-completion commutes with this filtered limit, we get that $X \to (\mathbb{F}_p)_\infty X$ Is a mod p homology isomorphism. Note that if $X \to Y$ is a homology isomorphism, then the Bousfield-Kan resolution is a levelwise weak equivalence

$$(\mathbb{F}_p)^\bullet X \to (\mathbb{F}_p)^\bullet Y$$

and so $(\mathbb{F}_p)_\infty X \xrightarrow{\sim} (\mathbb{F}_p)_\infty Y$. Thus, we get the diagonal map in the diagram

$$\begin{array}{ccc} X & \longrightarrow & (\mathbb{F}_p)_\infty X \\ \downarrow & \nearrow & \downarrow \sim \\ Y & \longrightarrow & (\mathbb{F}_p)_\infty Y. \end{array}$$

This proves that $(\mathbb{F}_p)_\infty$ is localization.