Lecture 21: Nilpotent spaces and $p$-completion

Paul VanKoughnett

November 19, 2014

For the record, the Sullivan conjecture is true for an arbitrary finite $p$-group $\pi$:

$$(F_p)\infty (X^\pi) \simeq ((F_p)\infty X)^{h\pi}.$$  

The very first fact you learn about $p$-groups is that they have a nontrivial center. Let $x \in Z(\pi)$ be a central element, and let $\sigma$ be the subgroup it generates. This is a normal subgroup! So we have a smaller $p$-group $\pi' = \pi/\sigma$; also, $X^\pi = (X^\sigma)^{\pi'}$, and likewise for homotopy orbits. So we get

$$(F_p)\infty (X^\pi) = ((F_p)\infty X)^{h\sigma} = ((F_p)\infty X)^{h\pi'},$$

and by induction on the order of $\pi$, this is

$$(F_p)\infty (X^\sigma)^{h\pi'} = (((F_p)\infty X)^{h\sigma})^{h\pi'} = ((F_p)\infty X)^{h\pi}. $$

So there’s really nothing in this more complicated case that requires anything.

In particular, if $\pi$ acts trivially, $(F_p)\infty X \simeq \text{map}(B\pi, (F_p)\infty X)$.

Let’s change tacks and talk some more about Bousfield-Kan $p$-completion.

**Definition 1.** Let $G$ be a group. $G$ is **nilpotent** if one of the following equivalent conditions holds:

1. The lower central series terminates;

2. $G$ has a finite filtration

   $$\{e\} \triangleleft G_n \triangleleft \cdots \triangleleft G_1 \triangleleft G$$

   (each $G_i$ normal in $G_{i-1}$) with $G_{i-1}/G_i$ abelian.

Any solvable group is nilpotent, but, for example, the free group on 2 generators, $\pi_1(S^1 \vee S^1)$, is not: its lower central series has associated graded the free Lie algebra on 2 generators, which is pretty big.

**Definition 2.** Let $G$ be a group and $M$ a $G$-module (an abelian group with an action of $G$). Then $M$ is **nilpotent** if it has a finite filtration by $G$-modules

$$0 \subseteq M_n \subseteq \cdots \subseteq M_1 \subseteq M$$

where $G$ acts trivially on $M_i/M_{i+1}$.

We can even generalize this to where $M$ is nonabelian; then a nilpotent group is a group that’s nilpotent for its own conjugation action.

**Definition 3.** A connected space is **nilpotent** if $\pi_1 X$ is nilpotent and acts nilpotently on $\pi_n X$, $n > 1$.

For example, any simply connected space is nilpotent. On the other hand, even if $\pi_1 X$ is abelian, $X$ might not be nilpotent.

**Example 4.** $S^1 \vee S^2$ is not a nilpotent space. Its universal cover is $\mathbb{R}$ with a copy of $S^2$ glued to each integer. Thus $\pi_2(S^1 \vee S^2) = \mathbb{Z}[\pi_1]$, where $\pi_1 \cong \mathbb{Z}$, say generated by $\tau$. If $f: \mathbb{Z}[\pi_1] \to M$ is a surjection onto a trivial $\pi_1$-module, then $\ker f$ contains $1 - \tau^i$ for all $i$, so $\ker(f)$ contains the augmentation ideal $I$ of $\mathbb{Z}[\pi_1]$. But this means that $M$ is a quotient of the integers, which are $\mathbb{Z}[\pi_1]/I$. In fact, $I$ is isomorphic to $\mathbb{Z}[\pi_1]$, via $x \mapsto x(1 - \tau) \in I$. So we won’t have the finite filtration we need.
Theorem 5. Let $X$ be a nilpotent space. Then $X \to (\mathbb{F}_p)_\infty X$ is the $H_* (\_; \mathbb{F}_p)$-localization of $X$, i.e. the terminal homology isomorphism out of $X$.

In other words, if $X \to Y$ is a mod $p$ homology isomorphism, then there’s a unique map $Y \to (\mathbb{F}_p)_\infty X$ such that the composition is $X \to (\mathbb{F}_p)_\infty X$. The existence of such a localization (for any homology theory, not just ordinary homology) was proved by Bousfield in an important paper in *Topology*, 1975.

Example 6 (Bousfield). $(S^1 \vee S^n) \to (\mathbb{F}_p)_\infty (S^1 \vee S^n)$, where $n \geq 2$, is not an isomorphism on mod $p$ homology. In fact, none of the maps

$$(\mathbb{F}_p)_\infty^k (S^1 \vee S^n) \to (\mathbb{F}_p)_\infty^{k+1} (S^1 \vee S^n)$$

are homology isomorphisms. Dwyer showed that you get a homology isomorphism if you do this transfinitely many times.

Proposition 7. Let $X$ be simply connected.

1. If $\pi_n X$ is finitely generated, then $\pi_n (\mathbb{F}_p)_\infty X = (\pi_n X)_p$.

2. More generally, there is a split short exact sequence

$$0 \to L_0 \pi_n X \to \pi_n (\mathbb{F}_p)_\infty X \to L_1 \pi_{n-1} X \to 0$$

where $L_i A$ is the $i$th left derived functor of completion.

You may have heard that completion is left exact. This only happens under finite type hypotheses – in general, the functor $L_0$ is not actually completion. Let’s digress to talk about this.

For an abelian group $A$, we have $A_p^\wedge = \lim (\mathbb{Z}/p^n \otimes A)$. In general, $\mathbb{Z}/p^n \otimes \cdot$ has left derived functors, and $\lim$ has right derived functors. So we have $L_1 A = \lim \text{Tor}_1 (\mathbb{Z}/p^n, A)$, and $L_0 A$ fits into an exact sequence

$$0 \to \lim (\mathbb{Z}/p^n \otimes A) \to L_0 A \to \lim \text{Tor}_1 (\mathbb{Z}/p^n, A) \to 0.$$ 

The higher derived functors are zero.

For example, let $\mathbb{Z}/p^\infty = \colim \mathbb{Z}/p^n \subset \mathbb{Q}/\mathbb{Z}$. One can show that $L_0 \mathbb{Z}/p^\infty = 0$, and that $L_1 \mathbb{Z}/p^\infty = \mathbb{Z}_p$. As a result, $(\mathbb{F}_p)_\infty K(\mathbb{Z}/p^\infty, n) = K(\mathbb{Z}_p, n+1)$. (These are good things to do in your chair when the lecture gets boring.)

Lemma 8 (Nilpotent fiber lemma). Let $p: E \to B$ be a fibration with $E$, $B$ connected and $\pi_1 E \to \pi_1 B$ onto. If $\pi_1 B$ acts nilpotently on $H_* (F, \mathbb{F}_p)$, then the natural map

$$(\mathbb{F}_p)_\infty F \to \text{fiber} (\mathbb{F}_p)_\infty E \to (\mathbb{F}_p)_\infty B$$

is a weak equivalence. That is, $(\mathbb{F}_p)_\infty$ preserves this fibration.

We can use this to study the $p$-completion functor on nilpotent spaces, by working up the Postnikov tower.

Lemma 9. Let $A$ be an abelian group. Then

$$K(A, n) \to (\mathbb{F}_p)_\infty K(A, n)$$

is a homology isomorphism, and

$$\pi_n (\mathbb{F}_p)_\infty K(A, n) = L_0 A, \quad \pi_{n+1} (\mathbb{F}_p)_\infty K(A, n) = L_1 A.$$

Proof: You can prove this by direct calculation using the Bousfield-Kan spectral sequence. Bousfield has a slicker way of doing it by defining a universal property for the right-hand side.
If $X$ is nilpotent, then we can write $X = \holim X_n$, where $\{X_n\}$ is the ‘refined’ Postnikov tower, a tower of fibrations

$$\cdots \to X_2 \to X_1 \to s = X_0$$

where each $X_n \to X_{n-1}$ has fiber $K(A, s_n)$ and $\pi_1 X_{n-1}$ acts trivially on $H_* K(A, s_n)$. (Note that $s \neq n$ in general: we need to filter each of the homotopy groups of $X$ into pieces with a trivial $\pi_1$-action.) Moreover, $s_n$ is increasing, and hits each integer $k$ only finitely many times.

By induction and the nilpotent fiber lemma, we get that

$$X_n \to (\mathbb{F}_p)_\infty X_n$$

is a mod $p$ homology isomorphism. After showing that $p$-completion commutes with this filtered limit, we get that $X \to (\mathbb{F}_p)_\infty X$ is a mod $p$ homology isomorphism. Note that if $X \to Y$ is a homology isomorphism, then the Bousfield-Kan resolution is a levelwise weak equivalence

$$(\mathbb{F}_p)^\bullet X \to (\mathbb{F}_p)^\bullet Y$$

and so $(\mathbb{F}_p)_\infty X \sim (\mathbb{F}_p)_\infty Y$. Thus, we get the diagonal map in the diagram

$$\begin{array}{ccc}
      X & \to & (\mathbb{F}_p)_\infty X \\
      \downarrow & \sim & \downarrow \\
      Y & \to & (\mathbb{F}_p)_\infty Y.
\end{array}$$

This proves that $(\mathbb{F}_p)_\infty$ is localization.