Lecture 22: p-compact groups

Paul VanKoughnett

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Question: Let $p > 2$ and $A = \mathbb{F}_p[x_1, \ldots, x_n] \in \mathcal{K}$. Is there a space X such that $H^*X \cong A$? If there is one, how many are there?

Note that if X is such a space and X is simply connected, $H^*\Omega X = \Lambda(y_1, \ldots, y_n)$. So we have a related question:

Question′: Classify all loop spaces ΩX with $H^*\Omega X$ finite.

This is the beginning of the p-compact group story. A **(Dwyer-Wilkerson) p-compact group** is a p-complete space X so that ΩX is connected and $H^*\Omega X$ is finite. (The 'group' is the space, not its loop space: the classifying space of a Lie group, rather than the Lie group itself. Confusing.) The problem of classifying p-compact groups has been solved, fairly recently in fact – it's not well-known in the US, but it's a popular subject in Europe.

Example 1. $(BG)_p = (\mathbb{F}_p)_\infty BG$ where G is a 1-connected Lie group.

Example 2. $BU(n)$, with $H^*BU(n) \cong \mathbb{F}_p[c_1, \ldots, c_n]$. In fact, this has more structure than just this cohomology ring: there's a map

$$(\mathbb{C}P^\infty)^{\times n} \cong BU(1)^{\times n} \to BU(n)$$

with the property $H^*BU(n) \xrightarrow{\cong} H^*(BU(1)^\times n)^{\Sigma_n}$. In fact, this comes out of the Lie group structure of $U(n)$: $U(1)^{\times n}$ is a maximal torus $T \subseteq U(n)$, and Σ_n is the Weyl group $N(T)/T$.

Question: Given a p-compact group X, can you produce a maximal torus

$$(BU(1)^\times_n)_p \to X_p?$$

A Weyl group? An element of order p, i. e. a map $B\mathbb{Z}/p \to X$? To do these things in Lie group theory, you need to actually use analysis. Here, surprisingly, we can use the T-functor.

Here's a toy case: $(\mathbb{C}P^\infty)_p = (BU(1))_p = K(\mathbb{Z}_p, 2)$. $\mathbb{Z}_p^\times$ contains a copy of $\mathbb{F}_p^\times = C_{p-1}$, which acts on $K(\mathbb{Z}_p, 2)$ (since $K(G, n)$ is functorial in G). Let Y be the homotopy orbits $EC_{p-1} \times_{C_{p-1}} K(\mathbb{Z}_p, 2)$. We have

$$H^*(Y, \mathbb{F}_p) = (H^*\mathbb{C}P^\infty)^{C_{p-1}} = (\mathbb{F}_p[x])^{C_{p-1}} = \mathbb{F}_p[y]$$

where $|y| = 2(p-1)$, $y = x^{p-1}$. (The action is just via $C_{p-1} \cong \mathbb{F}_p^\times$ multiplying on x.) This is an exclusively p-complete thing: C_{p-1} doesn't act on the integers. Y isn't simply connected, so we have to p-complete again. $Z = Y_p$ is simply connected and has $\Omega Z \simeq S_p^{2p-3}$, meaning that this p-complete sphere has an H-space structure.

Let's reverse engineer this. Let X be a p-complete space with $H^*X \cong \mathbb{F}_p[y]$, $|y| = 2(p-1)$. The Steenrod algebra structure is forced by $\mathcal{P}^{p-1}(y) = y^p$. In fact, there's an Adem relation

$$\underbrace{\mathcal{P}^1 \cdots \mathcal{P}^1}_{i} = \binom{p-i}{i} \mathcal{P}^i$$

for $0 \le i \le p-1$, so $\mathcal{P}^i(y) = \binom{p-i}{i} y^{1+i}$, and $\mathcal{P}^i(y) = 0$ for $i > p-1$. We get a map in $\mathcal{K}$, $H^*X \to \mathbb{F}_p[x] = H^*\mathbb{C}P^\infty$, with $y \mapsto x^{p-1}$. Algebraically, we've adjoined a $(p-1)$st root of y – that is, $\mathbb{F}_p[x] = (H^*X)[z]/(z^{p-1} - y)$. The roots of $z^{p-1} - y$ are ax, $a \in \mathbb{F}_p^\times$.

Now let's calculate $TH^*X = T\mathbb{F}_p[y] = (T\mathbb{F}_p[x])^{C_{p-1}}$. But $TH^*\mathbb{C}P^\infty = H^* \operatorname{map}(B\mathbb{Z}/p, \mathbb{C}P^\infty) = H^*(\mathbb{Z}/p \times \mathbb{C}P^\infty)$ – we can calculate this fairly easily using that $\mathbb{C}P^\infty$ is an Eilenberg-Mac Lane space.

$$TH^*\mathbb{C}P^\infty = (\mathbb{F}_p[x])^{\times \mathbb{Z}/p}.$$

The group C_{p-1} acts on the $\mathbb{F}_p[x]$ factor as well as, by multiplication, on the exponent. So

$$T\mathbb{F}_p[y] \cong \mathbb{F}_p[x]^{C_{p-1}} \times (\mathbb{F}_p[x]^{\times \mathbb{F}_p^\times})^{C_{p-1}} \cong \mathbb{F}_p[y] \times \mathbb{F}_p[x].$$

This is the cohomology of $X \sqcup \mathbb{C}P^\infty$. By Lannes' theorem,

$$X \sqcup (\mathbb{C}P^\infty)_p \ \widetilde{\text{map}}\ (B\mathbb{Z}/p, X).$$

This has two components, one of which, $T^\phi H^*X \cong H^*\mathbb{C}P^\infty$, corresponds to a map $\phi : H^*X \to H^*\mathbb{C}P^\infty \subseteq H^*B\mathbb{Z}/p$, and the composite

$$(\mathbb{C}P^\infty)_p \to \text{map}(B\mathbb{Z}/p, X) \to X,$$

where the right-hand map is evaluation at the basepoint, realizes ϕ. This is the p-compact maximal torus.

If this map can be made C_{p-1}-equivariant, we get an equivalence

$$(EC_{p-1} \times_{C_{p-1}} (\mathbb{C}P^\infty)_p)_p \xrightarrow{\sim} X.$$

The last step, then, is to study automorphisms of $B = (\mathbb{C}P^\infty)_p$ over X. That is, we should study the spaces of maps

$$\begin{array}{ccc} \text{map}_X(B,B) & \longrightarrow & \text{map}(B,B) \\ \downarrow & \lrcorner & \downarrow \\ \{\phi\} & \longrightarrow & \text{map}(B,X). \end{array}$$

We want to find a lift in the diagram

$$\begin{array}{ccc} & & \text{map}_X(B,B) \\ & \nearrow & \downarrow \\ C_{p-1} & \longrightarrow & [B,B]_{/[\phi]}. \end{array}$$

So far, we've done a similar analysis for the simpler diagram

$$\begin{array}{ccc} \text{map}(B\mathbb{Z}/p,B) & \longrightarrow & \text{map}(B\mathbb{Z}/p,B) \\ \downarrow & \lrcorner & \downarrow \\ \{\phi \circ i\} & \longrightarrow & \text{map}(B\mathbb{Z}/p,X), \end{array}$$

where $i : B\mathbb{Z}/p \to (\mathbb{C}P^\infty)_p$.

This is the program from now on. We'll take $A = \mathbb{F}_p[x_1, \dots, x_n]$, with $|x_i| = 2d_i$ and $p \nmid d_1 \cdots d_n$.

Theorem 3 (Adams-Wilkerson)**.** *There is a map $A \to H^*((BS^1)^{\times n}) = \mathbb{F}_p[t_1, \dots, t_n]$, and a finite group $W \subseteq GL_n(\mathbb{Z}_p)$ of order $d_1 \cdots d_n$, with $A = \mathbb{F}_p[t_1, \dots, t_n]^W$.*

Theorem 4 (Clark-Ewing)**.** *We won't prove this theorem, but it classifies such W, called **p-adic reflection groups**, and proves that A is actually the cohomology of a space, namely $EW \times_W ((BS^1)^{\times n})_p$.*

Theorem 5 (Dwyer-Miller-Wilkerson)**.** *They proved uniqueness: if $H^*X \cong A$, then it has a maximal torus and a W-action.*