## Lecture 23: The Dwyer-Miller-Wilkerson theorem

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November 24, 2014

**Theorem 1** (Adams-Wilkerson). Let  $A = \mathbb{F}_p[x_1, \ldots, x_n] \in \mathcal{K}$ , with  $|x_i| = 2d_i$  and  $p \nmid 2d_1 \cdots d_n$ . There is an embedding in  $\mathcal{K}$ ,  $A \to B = \mathbb{F}_p[t_1, \ldots, t_n]$ , with  $|t_i| = 2$ , such that if  $W = \operatorname{Aut}_{A/\mathcal{K}}(B)$ , then  $A = B^W$ .

Idea of proof. Let  $A \to B_{\infty}$  be the separable closure of A in the category of graded algebras. Let  $B \subseteq B_{\infty}$  be the maximal subextension  $A \subseteq B$  in  $\mathcal{K}$ . (This will turn out to equal  $B_{\infty}$ , though that won't be clear for a while. Define  $Q_0 = \mathcal{P}^1$ ,  $Q_n = [\mathcal{P}^{p^n}, Q_{n-1}]$ . These are "derivations" in  $\mathcal{A}$ .

**Lemma 2.** There are elements  $c_i \in A$ ,  $c_i \neq 0$ , so that

$$c_0 Q_0 + \dots + c_n Q_n = 0$$

 $on \ B.$ 

Let  $f(x) = c_0 x + c_1 x^p + \dots + c_n x^{p^n} \in A[x]$ . One shows that this splits over B, and the subalgebra generated by the roots is  $\mathbb{F}_p[t_1, \dots, t_n] \subseteq B$ . Then you have to show that  $\mathbb{F}_p[t_1, \dots, t_n] = B = B_{\infty}$ .  $\Box$ 

Remark 3. A priori,  $W \subseteq GL_n(\mathbb{F}_p)$ , but in fact, it lifts to  $GL_n(\mathbb{Z}_p)$ . It is called a *p*-adic reflection group, since for  $g \in W$ , rank $(g-1) \leq 1$ .

Note that  $H^*(EW \times_W \widehat{BT}) \cong B^W \cong A$ , where  $\widehat{BT} = (\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty)_p = K(\mathbb{Z}_p^n, 2).$ 

**Theorem 4** (Dwyer-Miller-Wilkerson). Let X be p-complete,  $H^*X \cong A$ . Then there is a maximal torus  $f: \widehat{BT} \to X$  realizing the inclusion  $A \subseteq B$ . If  $\operatorname{map}_X^+(\widehat{BT}, \widehat{BT})$  is the 'Weyl space' of self-equivalences of  $\widehat{BT}$  over X, then  $\operatorname{map}_X^+(\widehat{BT}, \widehat{BT})$  is discrete and isomorphic to the Weyl group W.

**Corollary 5.** The map  $\widehat{BT} \to X$  can be made W-equivariant, and induces a weak equivalence  $(EW \times_W \widehat{BT})_p \xrightarrow{\sim} X$ .

The map f can be produced using Lannes theory, which is why we're even talking about this theorem.

**Lemma 6.** Let  $V_1$  be a finite-dimension  $\mathbb{F}_p$ -vector space, and  $V_2 = H_2 \widehat{BT} \cong \mathbb{F}_p^n$ . Then

$$T_{V_1}H^*\widehat{BT}\cong H^*\widehat{BT}\otimes \mathbb{F}_n^{\mathrm{Hom}(V_1,V_2)}$$

with the second factor in degree zero.

*Proof.* If  $V_1 = \mathbb{F}_p^k$ , then  $T_{V_1} = T^{\circ k}$ . Also,  $H_*\widehat{BT} \cong \mathbb{F}_p[t_1] \otimes \cdots \otimes \mathbb{F}_p[t_n]$ . Thus,

$$T(H^*BT) \cong T(\mathbb{F}_p[t_1] \otimes \cdots \otimes \mathbb{F}_p[t_n])$$
  
$$\cong T(\mathbb{F}_p[t_1]) \otimes \cdots \otimes T(\mathbb{F}_p[t_n])$$
  
$$\cong (\mathbb{F}_p[t_1] \otimes \mathbb{F}_p^{\mathbb{F}_p}) \otimes \cdots \otimes (\mathbb{F}_p[t_n] \otimes \mathbb{F}_p^{\mathbb{F}_p})$$
  
$$\cong H^*\widehat{BT} \otimes \mathbb{F}_p^{\mathrm{Hom}(\mathbb{F}_p, V_2)}.$$

By induction, we get  $T_{\mathbb{F}_p^k}(H^*\widehat{BT}) = TH^*\widehat{BT} \otimes \mathbb{F}_p^{\operatorname{Hom}(\mathbb{F}_p^k, V_2)}$ .

**Theorem 7.** Let  $i: H^*X = A \to B = H^*\widehat{BT}$  be the inclusion above, and let  $V = V_1 = V_2$ . Then  $T_V^1H^*X$  (the component corresponding to the identity) is isomorphic to  $H^*\widehat{BT}$ .

Proof.

$$T_V H^* X = (T_V H^* \widehat{BT})^W = (H^* \widehat{BT} \otimes \mathbb{F}_n^{\mathrm{Hom}(V,V)})^W$$

W acts on  $H^*\widehat{BT}$  and  $\operatorname{Hom}(V,V) \supseteq GL_n(\mathbb{F}_p)$ . Thew orbit of the identity is a free *W*-orbit, so the corresponding component is  $H^*\widehat{BT}$ . (Meanwhile, if we'd taken the orbit of zero, we'd have gotten  $H^*X$  as the fixed points).

Finally, by Lannes theory and the above theorem, we get  $f: \widehat{BT} \to X$ .

## Theorem 8.

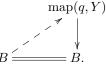
$$\operatorname{map}_X^+(\widehat{BT},\widehat{BT})\simeq W.$$

*Proof.* We want to bring this down to a calculation of  $\operatorname{map}_X(BV, \widehat{BT})$ , using the inclusion  $BV \to \widehat{BT}$  which is the fiber of  $Bp: \widehat{BT} \to \widehat{BT}$ ; we can then calculate this using the *T*-functor.

**Step 1:** Let  $q: E \to B$  be any fibration, and Y any space. Then there is a fibration

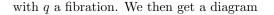
$$\dot{q}: \operatorname{map}(q, Y) \to B$$

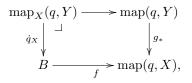
such that the space of sections  $R\Gamma(\dot{q})$  is weakly equivalent to map(E, Y). This is the space of maps of the form



By mumbo-jumbo, the fiber of  $\dot{q}$  is map(F, Y).

Step 2: You can relativize this. Start with a diagram





 $E \xrightarrow{f} X$ 

used to define map<sub>X</sub>(q, Y). Moreover,  $R\Gamma(\dot{q}_X) = \max_X(E, Y)$ , the space of diagrams



The fiber of  $\dot{q}_X$  is map<sub>X</sub>(F, Y).

**Step 3:** Specialize to the case where  $f = g : \widehat{BT} \to X$ , and  $q = Bp : \widehat{BT} \to \widehat{BT}$ , with fiber F = BV. Then  $\dot{q}_X$  is a map map<sub>X</sub> $(Bp, BT) \to \widehat{BT}$ , and  $R\Gamma(\dot{q}_X) = \max_X(\widehat{BT}, \widehat{BT})$ . The fiber of  $\dot{q}_X$  is map<sub>X</sub> $(BV, \widehat{BT})$ . **Step 4:** We're still not done because we haven't restricted to self-equivalences. Let  $\operatorname{map}_X^+(\widehat{BT}, \widehat{BT})$  be the space of weak equivalences over X, and in general, let  $\operatorname{map}_X^+(\cdot, \cdot)$  be the space of maps which are monomorphisms on cohomology. Then  $\dot{q}_X^+ : \operatorname{map}_X^+(\widehat{BT}, \widehat{BT}) \to \widehat{BT}$  has  $R\Gamma(\dot{q}_X^+) = \operatorname{map}_X^+(\widehat{BT}, \widehat{BT})$ , and the fiber is  $\operatorname{map}_X^+(BV, \widehat{BT})$ .

**Step 5:** Show  $\operatorname{map}_X^+(BV, \widehat{BT}) \cong W$ . Since  $\widehat{BT}$  is simply connected (it's  $K(\mathbb{Z}_p^n, 2))$ , by covering space theory,  $\operatorname{map}_X^+(Bp, \widehat{BT}) = W \times \widehat{BT} \to \widehat{BT}$ , and its space of sections is W. This proves the theorem.  $\Box$