# Lecture 23: The Dwyer-Miller-Wilkerson theorem

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**Theorem 1** (Adams-Wilkerson)**.** *Let $A = \mathbb{F}_p[x_1, \ldots, x_n] \in \mathcal{K}$, with $|x_i| = 2d_i$ and $p \nmid 2d_1 \cdots d_n$. There is an embedding in $\mathcal{K}$, $A \to B = \mathbb{F}_p[t_1, \ldots, t_n]$, with $|t_i| = 2$, such that if $W = \mathrm{Aut}_{A/\mathcal{K}}(B)$, then $A = B^W$.*

*Idea of proof.* Let $A \to B_\infty$ be the separable closure of $A$ in the category of graded algebras. Let $B \subseteq B_\infty$ be the maximal subextension $A \subseteq B$ in $\mathcal{K}$. (This will turn out to equal $B_\infty$, though that won't be clear for a while. Define $Q_0 = \mathcal{P}^1$, $Q_n = [\mathcal{P}^{p^n}, Q_{n-1}]$. These are "derivations" in $\mathcal{A}$.

**Lemma 2.** *There are elements $c_i \in A$, $c_i \neq 0$, so that*

$$c_0 Q_0 + \cdots + c_n Q_n = 0$$

*on $B$.*

Let $f(x) = c_0 x + c_1 x^p + \cdots + c_n x^{p^n} \in A[x]$. One shows that this splits over $B$, and the subalgebra generated by the roots is $\mathbb{F}_p[t_1, \ldots, t_n] \subseteq B$. Then you have to show that $\mathbb{F}_p[t_1, \ldots, t_n] = B = B_\infty$. □

*Remark* 3. A priori, $W \subseteq GL_n(\mathbb{F}_p)$, but in fact, it lifts to $GL_n(\mathbb{Z}_p)$. It is called a **$p$-adic reflection group**, since for $g \in W$, $\mathrm{rank}(g-1) \leq 1$.

Note that $H^*(EW \times_W \widehat{BT}) \cong B^W \cong A$, where $\widehat{BT} = (\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty)_p = K(\mathbb{Z}_p^n, 2)$.

**Theorem 4** (Dwyer-Miller-Wilkerson)**.** *Let $X$ be $p$-complete, $H^*X \cong A$. Then there is a maximal torus $f : \widehat{BT} \to X$ realizing the inclusion $A \subseteq B$. If $\mathrm{map}^+_X(\widehat{BT}, \widehat{BT})$ is the 'Weyl space' of self-equivalences of $\widehat{BT}$ over $X$, then $\mathrm{map}^+_X(\widehat{BT}, \widehat{BT})$ is discrete and isomorphic to the Weyl group $W$.*

**Corollary 5.** *The map $\widehat{BT} \to X$ can be made $W$-equivariant, and induces a weak equivalence $(EW \times_W \widehat{BT})_p \xrightarrow{\sim} X$.*

The map $f$ can be produced using Lannes theory, which is why we're even talking about this theorem.

**Lemma 6.** *Let $V_1$ be a finite-dimension $\mathbb{F}_p$-vector space, and $V_2 = H_2 \widehat{BT} \cong \mathbb{F}_p^n$. Then*

$$T_{V_1} H^* \widehat{BT} \cong H^* \widehat{BT} \otimes \mathbb{F}_p^{\mathrm{Hom}(V_1, V_2)},$$

*with the second factor in degree zero.*

*Proof.* If $V_1 = \mathbb{F}_p^k$, then $T_{V_1} = T^{\circ k}$. Also, $H_* \widehat{BT} \cong \mathbb{F}_p[t_1] \otimes \cdots \otimes \mathbb{F}_p[t_n]$. Thus,

$$\begin{aligned} T(H^* \widehat{BT}) &\cong T(\mathbb{F}_p[t_1] \otimes \cdots \otimes \mathbb{F}_p[t_n]) \\ &\cong T(\mathbb{F}_p[t_1]) \otimes \cdots \otimes T(\mathbb{F}_p[t_n]) \\ &\cong (\mathbb{F}_p[t_1] \otimes \mathbb{F}_p^{\mathbb{F}_p}) \otimes \cdots \otimes (\mathbb{F}_p[t_n] \otimes \mathbb{F}_p^{\mathbb{F}_p}) \\ &\cong H^* \widehat{BT} \otimes \mathbb{F}_p^{\mathrm{Hom}(\mathbb{F}_p, V_2)}. \end{aligned}$$

By induction, we get $T_{\mathbb{F}_p^k}(H^* \widehat{BT}) = T H^* \widehat{BT} \otimes \mathbb{F}_p^{\mathrm{Hom}(\mathbb{F}_p^k, V_2)}$. □

**Theorem 7.** *Let $i : H^*X = A \to B = H^*\widehat{BT}$ be the inclusion above, and let $V = V_1 = V_2$. Then $T_V^1 H^*X$ (the component corresponding to the identity) is isomorphic to $H^*\widehat{BT}$.*

*Proof.*

$$T_V H^*X = (T_V H^*\widehat{BT})^W = (H^*\widehat{BT} \otimes \mathbb{F}_p^{\mathrm{Hom}(V,V)})^W.$$

$W$ acts on $H^*\widehat{BT}$ and $\mathrm{Hom}(V,V) \supseteq GL_n(\mathbb{F}_p)$. Thew orbit of the identity is a free $W$-orbit, so the corresponding component is $H^*\widehat{BT}$. (Meanwhile, if we'd taken the orbit of zero, we'd have gotten $H^*X$ as the fixed points). ☐

Finally, by Lannes theory and the above theorem, we get $f : \widehat{BT} \to X$.

**Theorem 8.**

$$\mathrm{map}_X^+(\widehat{BT}, \widehat{BT}) \simeq W.$$

*Proof.* We want to bring this down to a calculation of $\mathrm{map}_X(BV, \widehat{BT})$, using the inclusion $BV \to \widehat{BT}$ which is the fiber of $Bp : \widehat{BT} \to \widehat{BT}$; we can then calculate this using the $T$-functor.

**Step 1:** Let $q : E \to B$ be any fibration, and $Y$ any space. Then there is a fibration

$$\dot{q} : \mathrm{map}(q, Y) \to B$$

such that the space of sections $R\Gamma(\dot{q})$ is weakly equivalent to $\mathrm{map}(E, Y)$. This is the space of maps of the form

$$\begin{array}{ccc} & & \mathrm{map}(q,Y) \\ & \nearrow & \downarrow \\ B & = & B. \end{array}$$

By mumbo-jumbo, the fiber of $\dot{q}$ is $\mathrm{map}(F, Y)$.

**Step 2:** You can relativize this. Start with a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ E & \xrightarrow{f} & X \\ {\scriptstyle q}\downarrow & & \\ B & & \end{array}$$

with $q$ a fibration. We then get a diagram

$$\begin{array}{ccc} \mathrm{map}_X(q,Y) & \longrightarrow & \mathrm{map}(q,Y) \\ {\scriptstyle \dot{q}_X}\downarrow & \lrcorner & \downarrow{\scriptstyle g_*} \\ B & \xrightarrow{f} & \mathrm{map}(q,X), \end{array}$$

used to define $\mathrm{map}_X(q, Y)$. Moreover, $R\Gamma(\dot{q}_X) = \mathrm{map}_X(E, Y)$, the space of diagrams

$$\begin{array}{ccc} & & Y \\ & \nearrow & \downarrow{\scriptstyle g} \\ E & \xrightarrow{f} & X. \end{array}$$

The fiber of $\dot{q}_X$ is $\mathrm{map}_X(F, Y)$.

**Step 3:** Specialize to the case where $f = g : \widehat{BT} \to X$, and $q = Bp : \widehat{BT} \to \widehat{BT}$, with fiber $F = BV$. Then $\dot{q}_X$ is a map $\mathrm{map}_X(Bp, BT) \to \widehat{BT}$, and $R\Gamma(\dot{q}_X) = \mathrm{map}_X(\widehat{BT}, \widehat{BT})$. The fiber of $\dot{q}_X$ is $\mathrm{map}_X(BV, \widehat{BT})$.

**Step 4:** We're still not done because we haven't restricted to self-equivalences. Let $\text{map}_X^+(\widehat{BT}, \widehat{BT})$ be the space of weak equivalences over $X$, and in general, let $\text{map}_X^+(\cdot, \cdot)$ be the space of maps which are monomorphisms on cohomology. Then $\dot{q}_X^+ : \text{map}_X^+(\widehat{BT}, \widehat{BT}) \to \widehat{BT}$ has $R\Gamma(\dot{q}_X^+) = \text{map}_X^+(\widehat{BT}, \widehat{BT})$, and the fiber is $\text{map}_X^+(BV, \widehat{BT})$.

**Step 5:** Show $\text{map}_X^+(BV, \widehat{BT}) \cong W$. Since $\widehat{BT}$ is simply connected (it's $K(\mathbb{Z}_p^n, 2)$), by covering space theory, $\text{map}_X^+(Bp, \widehat{BT}) = W \times \widehat{BT} \to \widehat{BT}$, and its space of sections is $W$. This proves the theorem. $\square$