Lecture 3: Odd primes and homological algebra

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A few words about Steenrod operations at odd primes. Here we have a new operation, namely the **Bockstein**. THe short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

gives a long exact sequence in cohomology with

$$\cdots \to H^n(X;\mathbb{Z}/p^2) \to H^n(X;\mathbb{Z}/p) \xrightarrow{\beta} H^{n+1}(X;\mathbb{Z}/p) \to \cdots$$

where β is the Bockstein.

We can do this at $p = 2$, of course, but in this case $\beta = \mathrm{Sq}^1$. Note that $\beta^2 = 0$, at all primes. We also have a Leibniz rule

$$\beta(xy) = \beta(x)y + (-1)^{|x|}x\beta(y).$$

There are Steenrod operations

$$\mathcal{P}^i : H^n(X) = H^n(X;\mathbb{Z}/p) \to H^{n+2i(p-1)}(X).$$

This is the first place that you see the number $2(p-1)$, which is everywhere in topology. Again, these operations are subject to some axioms.

1. $\mathcal{P}^0(x) = x$ and $\mathcal{P}^i(x) = 0$ if $2i > |x|$.
2. $\mathcal{P}^n(x) = x^p$ if $|x| = 2n$. (These two are the unstable relations.)
3. $\mathcal{P}^i(xy) = \sum \mathcal{P}^j\, x \cdot \mathcal{P}^k\, y$.
4. There are Adem relations, which will remain unspoken.

Remark 1. Because of the Leibniz rule, $\beta(x^p) = px(\beta x)^{p-1} = 0$, when x is in even degree. We can generalize the other unstable relation to $\beta^\epsilon\, \mathcal{P}^i(x) = 0$ if $2i + \epsilon > n$, where $\epsilon = 0$ or 1.

The general Steenrod operation, then, can be written as

$$\mathcal{P}^I = \beta^{\epsilon_0}\, \mathcal{P}^{i_1}\, \beta^{\epsilon_1} \cdots \mathcal{P}^{i_s}\, \beta^{\epsilon_s}$$

where $i_t \geq 0$, $\epsilon = 0$ or 1. This is **admissible** if $i_t \geq pi_{t+1} + \epsilon_t$. The **excess** is

$$e(I) = 2i_1 + \epsilon_0 - \sum_{t>1} 2i_t(p-1) - \sum_{t>0} \epsilon_t.$$

Again, the admissible monomials form a basis for the Steenrod algebra. If $e(I) > n$, then $\mathcal{P}^I(x) = 0$ for $x \in H^n X$. Again, we have functors

$$\mathcal{K} \overset{U}{\rightleftarrows} \mathcal{U} \overset{\Omega^\infty}{\rightleftarrows} \mathsf{Mod}_{\mathcal{A}},$$

where $\mathcal{K}$ is the category of unstable algebras, $\mathcal{U}$ that of unstable modules, and $\mathsf{Mod}_{\mathcal{A}}$ that of all modules.

Example 2. Let C_p be the pth roots of unity. This is a subgroup of S^1, which acts on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$, and these actions are compatible with the inclusions $\mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$. Thus we can define

$$BC_p = \bigcup_n S^{2n+1}/C_p.$$

This has the properties that $\pi_1 BC_p = C_p$ and $\pi_n BC_p = 0$ for $n \geq 1$ (the universal cover is contractible). We have $H^* BC_p \cong E(x) \otimes \mathbb{F}_p[y]$, the tensor product of an exterior algebra generated in degree 1 and a polynomial algebra generated in degree 2. Almost the only thing that can happen does: $\beta x = y$, and $\mathcal{P}^1 y = y^p$ induces all the further Steenrod operations.

Homological algebra of unstable modules

Let $M \in \mathsf{Mod}_{\mathcal{A}}$. Then $\mathrm{Hom}_{\mathcal{A}}(\Sigma^n \mathcal{A}, M) \cong M^n$, the degree n elements of M. Here we have defined $(\Sigma^n N)^{k+n} = N^k$, so that $\Sigma \widetilde{H}^* X = \Sigma \widetilde{H}^* X$. $\Sigma^n A$, then, is the free module on one generator in degree n.

If $M \in \mathcal{U}$, then

$$\mathrm{Hom}_{\mathcal{A}}(\Sigma^n \mathcal{A}, M) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega^\infty \Sigma^n \mathcal{A}, M) \cong M^n.$$

Definition 3. We define $F(n) = \Omega^\infty \Sigma^n M$.

This is evidently projective. What does **projective** mean? It means that homming out of it preserves exact sequences, or equivalently that a diagram

$$\begin{array}{ccc} & & M \\ & \nearrow & \downarrow \\ F(n) & \longrightarrow & N \end{array}$$

always has the dotted arrow filling it in. So $F(n)$ is projective because $M \mapsto M^n$ is an exact functor.

The category $\mathcal{U}$ has **enough projectives**, meaning that every module has a surjection from a projective module. Indeed, just take the obvious maps

$$\bigoplus_n \bigoplus_{x \in M^n} F(n) \to M.$$

Having enough projectives means that we can form projective resolutions and thus define derived functors, like $\mathrm{Ext}^s_{\mathcal{U}}(M, N)$.

Exercise 4. (At $p = 2$) The elements $\mathrm{Sq}^I(i_n) \in F(n)$, where Sq^I is admissible, $e(I) \leq n$, and $i_n \in F(n)_n$ is the generator, form a basis.

This means that the $F(n)$ are never free: they have fewer nonzero Steenrod operations than the Steenrod algebra itself. For example, in $F(1)$, we only have the basis elements

$$\mathrm{Sq}^{2^k} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1(i_1).$$

Also note that $F(1)$ is not the cohomology of any space. If it were, then $\mathrm{Sq}^1(i_1)$ would have to be i_1^2, and $\mathrm{Sq}^2 \mathrm{Sq}^1(i_1) = i_1^4$, but $F(1)^3 = 0$, so i_1^3 would have to vanish. On the other hand,

$$H^*(\mathbb{R}P^\infty) = U(F(n)) = \mathrm{Sym}(F(1))/(\mathrm{Sq}^{|x|} x = x^2) \cong \mathbb{F}_2[i_1].$$

We've been stuck in the 50s so far, so let's enter the 80s.

Remark 5. We have an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}(F(n), M) &\cong M^n \\ f &\mapsto f(i_n) \end{aligned}$$

which means that the functor $M \mapsto F(n)$ is representable. Thus, the Yoneda lemma applies, saying that natural transformations $M^n \to M^k$ are in natural bijection with $\mathcal{U}$-module maps $F(k) \to F(n)$. In particular, if $\theta \in \mathcal{A}^k$ is a Steenrod operation, then $\theta : M^n \to M^{n+k}$ has to correspond to a map $F(n+k) \to F(n)$, which has to correspond to a degree $n+k$ element of $F(n)$, by the adjunction above. And this is just θi_n. What else could it be?

Theorem 6. *The functor*

$$\mathcal{U}^{\mathrm{op}} \to \mathbb{F}_p\text{--}\mathsf{VectorSpaces}$$

sending M^n to the dual vector space $(M^n)^*$ is representable. That is, there is a module $J(n) \in \mathcal{U}$ and a map $\phi_n : J(n)^n \to \mathbb{F}_p$ such that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}(M, J(n)) &\xrightarrow{\cong} (M^n)^* \\ f &\mapsto \phi_n \circ f \end{aligned}$$

is an isomorphism.

These are the **Brown-Gitler modules** – they were originally called $G(n)$ for Gitler, and then some French person misheard.

Since $M \to (M^n)^*$ is exact, the modules $J(n)$ are injective, and the category $\mathcal{U}$ has enough injectives (every module can be embedded into an injective).

Proof. If $J(n)$ exists at all, we have to have $J(n)^k = \mathrm{Hom}_{\mathcal{U}}(F(k), J(n)) = (F(k)^n)^*$. So we just define $J(n)$, as a graded vector space, to be the direct sum of $(F(k)^n)^*$ in degree n. If $\theta \in \mathcal{A}^s$, then we need a map

$$J(n)^k \xrightarrow{\theta} J(n)^{k+s},$$

that is,

$$(F(k)^n)^* \to (F(k+s)^n)^*,$$

and we can take this map to be the dual of $\theta i_k : F(k+s) \to F(k)$ restricted to degree n. This defines $J(n)$ as an unstable module.

Note that $J(n)^n = (F(n)^n)^* \cong \mathbb{F}_p^*$, defining a map $\phi_n : J(n)^n \to \mathbb{F}_p$. Thus we get the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}(M, J(n)) &\to (M^n)^* \\ f &\mapsto \phi_n \circ f \end{aligned}$$

This is an isomorphism if $M = F(k)$. Therefore, it's an isomorphism for all projective M, since these are a sum of $F(k)$ and both sides send sums to products. For general M, choose (the beginning of) a projective resolution

$$F_1 \to F_0 \to M \to 0.$$

We get

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathcal{U}}(F_1, J(n)) & \longleftarrow & \mathrm{Hom}_{\mathcal{U}}(F_0, J(n)) & \longleftarrow & \mathrm{Hom}_{\mathcal{U}}(M, J(n)) & \longleftarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ (F_1^n)^* & \longleftarrow & (F_0^n)^* & \longleftarrow & M & \longleftarrow & 0. \end{array}$$

By the five lemma, the right-hand vertical map is an isomorphism. (This proof is a special case of a general theorem called the Special Adjoint Functor Theorem.) □