Lecture 4: The Milnor basis

October 3, 2014

Last time we mentioned that the category $\mathcal{U}$ of unstable modules has both enough projectives enough injectives. A generating set of injectives was the modules $J(n)$ with

$$\mathrm{Hom}_{\mathcal{U}}(M, J(n)) = (M^n)^*,$$

which is plainly an exact functor of M. A generating set of projectives was the modules

$$F(n) = \Omega^\infty(\Sigma^n \mathcal{A}) = \{\mathrm{Sq}^I(i_n) : e(I) \leq n\}$$

(at $p = 2$) with

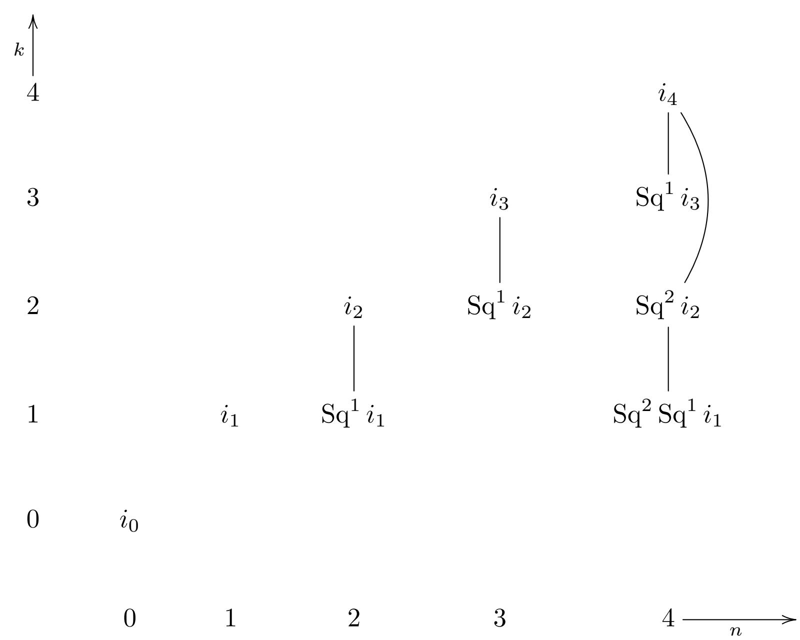
$$\mathrm{Hom}_{\mathcal{U}}(F(n), M) = M^n.$$

In particular,

$$J(n)^k \cong \mathrm{Hom}_{\mathcal{U}}(F(n), J(k)) = (F(n)^k)^*.$$

For $\theta \in \mathcal{A}^s$, there were induced maps $\theta i_k : F(k+s) \to F(k)$ and $\theta : J(n)^k \to J(n)^{k+s}$.

We can draw bases for the $F(k)$ and $J(n)$ simultaneously on a 2-dimensional grid. Each $J(n)$ occupies a column, each $F(k)$ a row. (The arrows are explained below.)



It remains to say how the Steenrod operations act. For example, we have $\mathrm{Sq}^1 : F(2) \to F(1)$, which sends i_2 to $\mathrm{Sq}^1 i_1$. (There's one of these maps for each element of $F(1)^2$.) On the $J(n)$, of course, these are acting on the dual vector spaces, so we could write $\mathrm{Sq}^1 : (\mathrm{Sq}^1 i_1)^* \mapsto (i_2)^*$. Thus, for instance, we also have $\mathrm{Sq}^1 : (\mathrm{Sq}^2 \mathrm{Sq}^1 i_1)^* \mapsto (\mathrm{Sq}^2 i_2)^*$. Once the Adem relations enter the picture, things get more complicated. It'd be nice to have a better way of doing this.

Remark 1. If M and N are unstable modules, so is $M \otimes N$ with the diagonal action

$$\mathrm{Sq}^i(x \otimes y) = \sum_{j+k=i} \mathrm{Sq}^j(x) \otimes \mathrm{Sq}^k(y).$$

Using this tensor product,

$$\widetilde{H}^*(X \otimes Y) \cong \widetilde{H}^*(X \wedge Y)$$

as unstable modules.

There are then maps

$$F(n) \to F(j) \otimes F(k), \qquad j + k = n,$$

with $i_n \mapsto i_j \otimes i_k$. Dually, we get maps $F(j)^* \otimes F(k)^* \to F(j+k)^*$ which are associative, commutative, and unital with unit $(i_0)^* \in F(0)^*$. Thus we get a bigraded commutative $\mathbb{F}_p$-algebra

$$J(*)^\bullet = F(\bullet)^*.$$

(Confusing point: $*$ is being used both to mean 'dual' and as one of the two gradings.)

Let $\xi_j \in (F(1)^{2^j})^*$ be dual to $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1(i_1)$, for $j \geq 0$.

Theorem 2 (Milnor, Miller)**.** *The resulting map*

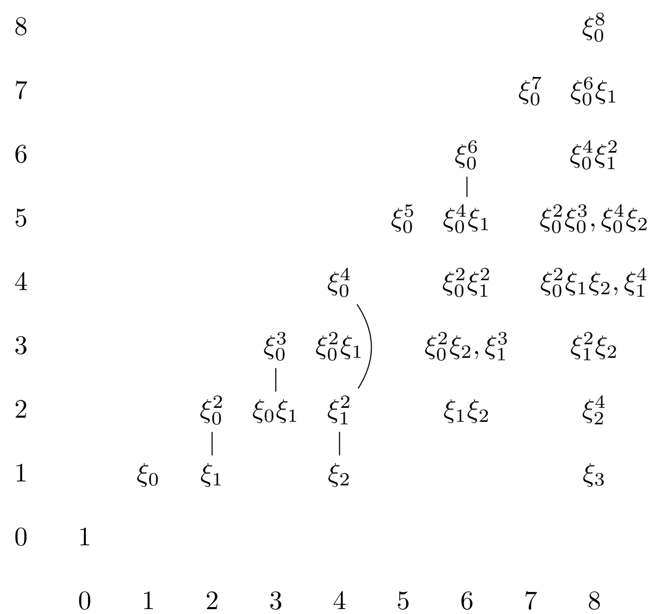
$$\mathbb{F}_2[\xi_0, \xi_1, \dots] \xrightarrow{\cong} F(\bullet)^*$$

is an isomorphism. The $\mathcal{A}$-action on the $\bullet$ variable is given by

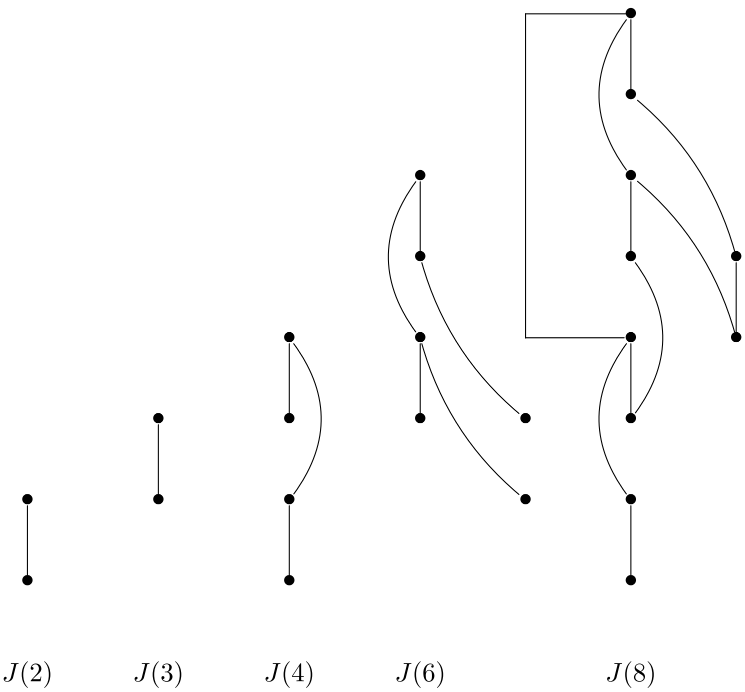
$$\mathrm{Sq}^1 \xi_i = \xi_{i-1}^2$$

if $i \geq 1$. (Since these modules are unstable, this describes the action completely.)

Here's a partial diagram of the first few $J(n)$. We'll pay less attention to the odd ones, for reasons that will become clear.



At this point, it's a little clearer to just do the cell diagrams. Here they are for the first few.



Remark 3. Milnor noticed that $J(8)$ can't be the cohomology of any space: the class ξ_0^8 would have to be an eighth power of something, which it can't be. It is a retract of a space, though, and a piece of it is isomorphic to the cohomology of $\mathbb{R}P^8$.

Remark 4. By construction, $\mathcal{A} = \lim_n \Sigma^{-n} F(n)$, so

$$\mathcal{A}^* \cong \operatorname{colim}_n \Sigma^{-n} F(n)^* \cong \mathbb{F}_2[\xi_0, \xi_1, \dots]/(\xi_0 = 1) \cong \mathbb{F}_2[\xi_1, \xi_2, \dots],$$

a more familiar statement of Milnor's result.

Sketch proof of the Milnor-Miller theorem. Let $J = (j_1, \dots, j_s)$ be admissible with $e(J) \le n$. Define $f(J) = (j_1 - 2j_2, j_2 - 2j_3, \dots, j_s)$ – since J is admissible, these are all positive numbers. Consider the map

$$\mathrm{Sq}^J(i_n) \leftrightarrow \xi_0^{n-e(J)} \xi_1^{j_1 - 2j_2} \cdots \xi_s^{j_s} =: \xi_0^{n-e(J)} \xi^{f(J)}.$$

We claim that this is a one-to-one correspondence between the standard basis for $\mathbb{F}_2[\xi_0, \xi_1, \dots]$ and the basis of $F(\bullet)^*$ given by the duals of the admissible sequences. So these have the same rank, and thus the map is an isomorphism.

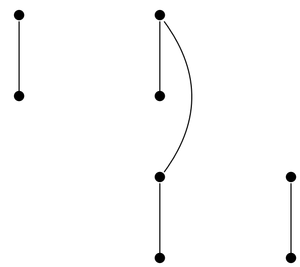
We just have to check that the above map is actually one-to-one. We just check that

$$\langle \xi_0^{n-e(J)} \xi^{f(J)}, \mathrm{Sq}^k(i_n) \rangle = \begin{cases} 1 & J = K \\ 0 & J > K \text{ in the lexicographic ordering.} \end{cases}$$

This is a hard calculation, so we won't actually do it. □

Observation 5. 1. It looks like $J(2n+1) \cong \Sigma J(2n)$ by multiplying by ξ_0.

2. It also looks like $\xi_0 : \Sigma J(2n-1) \to J(2n)$ is an injection with cokernel isomorphic to $J(n)$. See below.



$$\Sigma J(3) \longrightarrow J(4) \longrightarrow J(2)$$

Let's turn this observation into a theorem: you can read it off the algebra, but there's a more conceptual reason. There's a suspension functor $\Sigma : \mathcal{U} \to \mathcal{U}$ sending $M \mapsto \Sigma M$ with $(\Sigma M)^{n+1} = M^n$. It has a left adjoint, which we might as well call 'loops', with

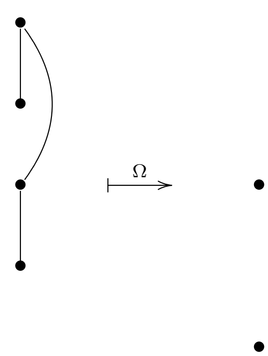
$$\mathrm{Hom}_{\mathcal{U}}(\Omega M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, \Sigma N).$$

(It's on the wrong side since we started with cohomology of spaces.) We could define ΩM by shifting everything down by 1, but this might not be an unstable module any longer. Instead, we have to define

$$(\Omega M)^n = M^{n+1}/\{\mathrm{Sq}^{|x|}(x)\},$$

after checking, of course, that the thing under the slash is an $\mathcal{A}$-submodule of M. In particular, the thing under the slash is in even degrees, so that $(\Omega M)^n = M^{n+1}$ if $n+1$ is odd.

Example 6.



The classes in degrees 4 and 2 are top-dimensional squares, so they die under Ω; the classes in degrees 3 and 1 survive one degree lower.

Proposition 7. $\Sigma J(2n) = J(2n+1)$.

Proof.

$$\mathrm{Hom}_{\mathcal{U}}(M, \Sigma J(2n)) = \mathrm{Hom}_{\mathcal{U}}(\Omega M, J(2n)) = ((\Omega M)^{2n})^* \tag{1}$$

$$= (M^{2n+1})^* = \mathrm{Hom}_{\mathcal{U}}(M, J(2n+1)) \tag{2}$$

By Yoneda's lemma, $\Sigma J(2n) = J(2n+1)$. □