Lecture 5: The doubling functor

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The goal of the next two classes is to show that $\widetilde{H}^*\mathbb{R}P^\infty$ (or $\widetilde{H}^*B\mathbb{Z}/p$, for $p > 2$) is injective in $\mathcal{U}$. This is a key part of the proof of the Sullivan conjecture.

Remark 1. We already have canonical injectives $J(n)$, so there's an inclusion $\widetilde{H}^*\mathbb{R}P^\infty \hookrightarrow \prod J(n_\alpha)$, but there's something a little weird about products (unclear – maybe that we want a map $\bigoplus J(n_\alpha) \to \widetilde{H}^*\mathbb{R}P^\infty$ splitting this, and can't find one, or maybe that there's just no obvious splitting of the given map). So we need to build more injectives.

Recall that $J(*)^\bullet = F(\bullet)^* \cong \mathbb{F}_2[\xi_0, \xi_1, \dots]$ with $\xi_i \in J(2^i)^1$. Define the **weight** of ξ_i to equal 2^i and the weight of a monomial $\xi_0^{i_0} \cdots \xi_n^{i_n} = i_0 2^0 + \cdots + i_n 2^n$ (so weight is the $*$ grading). Then $J(n)$ is spanned by monomials of weight n.

Example 2. $J(4)$ is generated by ξ_2, ξ_1^2, $\xi_0^2\xi_1$, and ξ_0^4.

Observation 3. If $n = a_0 + 2a_1 + \cdots + 2^k a_k$, where $a_i = 0$ or 1 (so this is the base 2 expansion of n), then the element of lowest degree in $J(n)$ is exactly $\xi_0^{a_0}\xi_1^{a_1}\cdots\xi_n^{a_n}$. In $J(4)$, the element of lowest degree is $\xi_2 \in J(4)^1$.

In particular, $J(2^k-1)^s = 0$ if $s < k$, because $2^k - 1 = 1 + \cdots + 2^{k-1}$, and the corresponding monomial $\xi_0\xi_1\cdots\xi_{k-1}$ has degree k.

At odd primes,

$$J(*)^\bullet \cong \mathbb{F}_p[\tau_{-1}, \xi_0, \xi_1, \dots] \otimes \Lambda(\tau_0, \tau_1, \dots)/(\tau_{-1}^2 = \xi_0)$$

(the right-hand factor is an exterior algebra on the given generators), where

$$\tau_i \in (F(1)^{2p^i})^* \text{ for } i \geq 0, \qquad \tau_{-1} \in (F(1)^1)^*, \qquad \xi_i \in (F(2)^{2p^i})^*.$$

The Steenrod algebra action can be deduced from $\beta\tau_i = \xi_i$, $\mathcal{P}^1\,\xi_i = \xi_{i-1}^p$.

Let's go back to $p = 2$.

Proposition 4. *$\Sigma J(2n-1) \cong J(2n)$, and $\Sigma J(2n) = \ker(\mathrm{Sq}^n : J(2n) \to J(n))$.*

This map $\cdot\,\mathrm{Sq}^n : J(2n) \to J(n)$ corresponds, by the Yoneda lemma, to the natural transformation

$$(\mathrm{Sq}^n)^* : \mathrm{Hom}(M, J(2n)) \cong (M^{2n})^* \to (M^n)^* \cong \mathrm{Hom}(M, J(n)).$$

Define $\Phi : \mathcal{U} \to \mathcal{U}$ to be the 'doubling functor' – 'if you put in the cohomology of $\mathbb{R}P^\infty$, you get out the cohomology of $\mathbb{C}P^\infty$.' Explicitly,

$$\Phi^n(M)^{2n} = M^n, \qquad \Phi^n(M)^{2n+1} = 0$$

as an $\mathbb{F}_2$-vector space. If $x \in M^n$, then write $\phi(x)$ for the corresponding element in $\Phi(M)^{2n}$. We define the Steenrod operations by

$$\mathrm{Sq}^{2i}\,\phi(x) = \phi(\mathrm{Sq}^i\,x), \qquad \mathrm{Sq}^{2i+1}\,\phi(x) = 0.$$

Exercise 5. Show that $\Phi(M)$ is actually a module over $\mathcal{A}$.

There's a map $\lambda = \lambda_M : \Phi^n(M) \to M$ given by $\phi(x) \mapsto \mathrm{Sq}^{|x|}\,x$.

Example 6. One easily observes that $\Phi(\widetilde{H}^*(\mathbb{R}P^4)) = \widetilde{H}^*(\mathbb{C}P^4)$. The map $\lambda : \widetilde{H}^*(\mathbb{C}P^4) \cong \mathbb{F}_2[x_2]/(x^5) \to \widetilde{H}^*(\mathbb{R}P^4) \cong \mathbb{F}_2[y_1]/(y^5)$ sends $x \mapsto y^2$ (and thus $x^2 \mapsto y^4 = \mathrm{Sq}^2(y)$). (CELL DIAGRAM)

Exercise 7. Show that λ_M is an $\mathcal{A}$-module map.

Note that the cokernel of λ_M is $M/\{\mathrm{Sq}^{|x|}(x)\} = \Sigma\Omega M$. But what's the kernel?

Remark 8. The functor $M \mapsto \Omega M$ is defined by a quotient, so it's right exact. This means that a short exact sequence

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

is sent to an exact sequence

$$\Omega M_0 \to \Omega M_1 \to \Omega M_2 \to 0.$$

By extending this leftwards to a long exact sequence, one gets the **left derived functors** of Ω, which for historical reasons are written $\Omega_s M$ for $s \geq 0$ (with $\Omega_0 = \Omega$).

The kernel of λ_M is going to be $\Sigma\Omega_1 M$.

Proposition 9. *Let $F : \mathcal{U} \to \mathbb{F}_p-\mathsf{VectorSpaces}$ be a right exact functor, and suppose we have a functor*

$$C_\bullet : \mathcal{U} \to \mathsf{Ch}_*(\mathbb{F}_p)$$

(the target is the category of chain complexes) such that

1. *there is a natural isomorphism $H_0 C_\bullet(M) \cong F(M)$,*
2. *$C_\bullet$ is exact,*
3. *and $H_s C_\bullet(P) = 0$ for P projective, $s > 0$.*

Then $H_s C_\bullet$ is naturally isomorphic to the sth left derived functor $L_s F$ of F.

Example 10. In our case, we put

$$C_\bullet(M) = \cdots \to 0 \to \Phi(M) \stackrel{\lambda_M}{\to} M.$$

Then $H_0 C_\bullet(M) = \Sigma\Omega M$, as we've seen, and $C_\bullet$ is exact since we can check this on the underlying graded vector spaces. Finally, if P is projective, then $\lambda_P : \Phi(P) \to P$ is injective: indeed, we only need to check this for $F(n)$, $n \geq 0$, in which

$$\phi(\mathrm{Sq}^{j_1} \cdots \mathrm{Sq}^{j_s}(i_n)) = \mathrm{Sq}^{j_1 + \cdots + j_s + n}\,\mathrm{Sq}^{j_1} \cdots \mathrm{Sq}^{j_s}(i_n).$$

If $(j_1, \ldots, j_s)$ is admissible and has excess $\leq n$, then $(j_1 + \cdots + j_s + n, j_1, \ldots, j_s)$ has excess exactly n and is admissible, proving the claim.

Thus, $H_1 C_\bullet(M) = \ker(\Phi(M) \to M) = \Sigma\Omega_1 M$, and $\Omega_s M = 0$ for $s > 1$.

Here's a sketch of the proof of Proposition 9. You need to construct a map $H_s C_\bullet(M) \to L_s F(M)$. Suppose given an exact sequence

$$0 \to K \to P \to M \to 0$$

with P projective. Then

$$0 \to C_\bullet K \to C_\bullet P \to C_\bullet M \to 0$$

is exact, and $H_s C_\bullet(P) = 0$ for $s > 0$. Thus, $H_{s+1} C_\bullet(M) \cong H_s C_\bullet(K)$ for $s \geq 1$. The map we want, on H_1, is constructed via the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_1 C_\bullet(M) & \longrightarrow & H_0 C_\bullet(K) & \longrightarrow & H_0 C_\bullet(P) & \longrightarrow & H_0 C_\bullet(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 F(M) & \longrightarrow & F(K) & \longrightarrow & F(P) & \longrightarrow & F(M) & \longrightarrow & 0.
\end{array}$$

in fact, all the solid vertical maps are isomorphisms, so this forces the map on H_1 to be an isomorphism. By induction, we get isomorphisms for all s.

Proof of Proposition 4. We've done the even part of this already. Now let's show that

$$0 \to \Sigma J(2n-1) \to J(2n) \stackrel{\cdot \operatorname{Sq}^n}{\to} J(n) \to 0$$

is exact. It suffices to show that this is exact after applying $\operatorname{Hom}_{\mathcal{U}}(F(k), \cdots)$ for all k. This gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}(F(k), \Sigma J(2n-1)) & \longrightarrow & \operatorname{Hom}(F(k), J(2n)) & \xrightarrow{\operatorname{Sq}^n} & \operatorname{Hom}(F(k), J(n)) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & (\Omega F(k)^{2n-1})^* & \longrightarrow & (F(k)^{2n})^* & \xrightarrow{\lambda^*} & (F(k)^n)^* & \longrightarrow & 0. \end{array}$$

But $(\Omega F(k)^{2n-1})^* \cong (\Sigma\Omega F(k)^{2n})^*$ since this is in odd degrees, and $(F(k)^n)^* \cong (\Phi F(k)^{2n})^*$, as we've checked. So this is the dual of the exact sequence

$$0 \to \Phi F(k) \stackrel{\lambda}{\to} F(k) \to \Sigma\Omega F(k) \to 0.$$

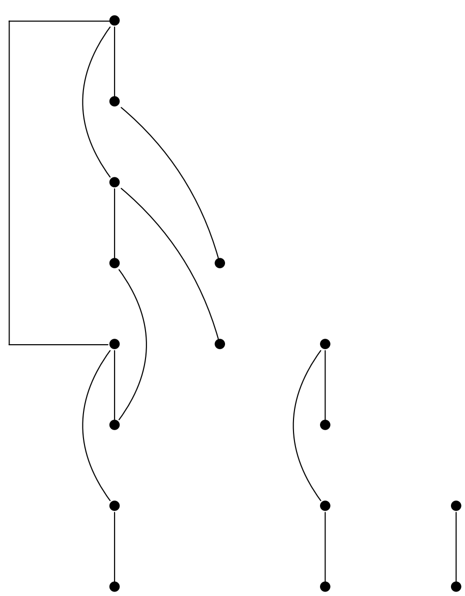
□

Note that $\Omega F(k) \cong F(k-1)$ by Yoneda:

$$\operatorname{Hom}_{\mathcal{U}}(\Omega F(k), M) \cong \operatorname{Hom}(F(k), \Sigma M) \cong (\Sigma M)^k \cong M^{k-1} \cong \operatorname{Hom}_{\mathcal{U}}(F(k-1), M).$$

Definition 11. $M \in \mathcal{U}$ is **reduced** if $\lambda : \Phi M \to M$ is injective.

For example, $F(k)$ and $\widetilde{H}^*(\mathbb{R}P^\infty)$ are reduced. If $x^i \in H^i\mathbb{R}P^\infty$, then $\operatorname{Sq}^i x^i = x^{2i} \neq 0$.



$$J(8) \xrightarrow[\cdot \operatorname{Sq}^4]{} J(4) \xrightarrow[\cdot \operatorname{Sq}^2]{} J(2)$$

Definition 12. Let $K(1) = \lim(\cdots \to J(8) \to J(4) \to J(2))$ where the map $J(2^{k+1}) \to J(2^k)$ is $\cdot \operatorname{Sq}^{2^k}$.

Then the nonzero maps $\widetilde{H}^*\mathbb{R}P^\infty \to J(2^k)$ (there's just one of these for each k) assemble to a map $\widetilde{H}^*\mathbb{R}P^\infty \to K(1)$.

Theorem 13 (Carlsson)**.** *The module $K(1)$ is a reduced injective, and the map $\widetilde{H}^*\mathbb{R}P^\infty \to K(1)$ is a split inclusion.*

Corollary 14. $\widetilde{H}^*\mathbb{R}P^\infty \in \mathcal{U}$ *is injective.*

Note that the finite $\widetilde{H}^*\mathbb{R}P^n$'s are not injective – for example, $\widetilde{H}^*\mathbb{R}P^8$ has a non-split inclusion into $J(8)$, as one can see from the above cell diagram.