Lecture 6: Carlsson's theorem

October 10, 2014

Today, we're going to prove and explore an amazing result: that $\widetilde{H}^*\mathbb{R}P^\infty$ (or $\widetilde{H}^*B\mathbb{Z}/p$, for $p > 2$) is an injective object in $\mathcal{U}$.

Remark 1. Don't take notes on this, but what does 'injective' mean? It means that there are arrows filling any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & M \\ & & \downarrow & \swarrow & \\ & & \widetilde{H}^*\mathbb{R}P^\infty. & & \end{array}$$

If you think this is easy, you should try to do it. There's just no method. What we need to do instead is to split $\widetilde{H}^*\mathbb{R}P^\infty$ out of something more obviously injective.

Definition 2. Let

$$K(n) = \lim\{\cdots \to J(4n) \overset{\cdot\,\mathrm{Sq}^{2n}}{\to} J(2n) \overset{\cdot\,\mathrm{Sq}^{n}}{\to} J(n)\},$$

where the maps fit into short exact sequences

$$0 \to \Sigma J(2n-1) \to J(2n) \overset{\cdot\,\mathrm{Sq}^{n}}{\to} J(n) \to 0.$$

Of course, $K(2n) \cong K(n)$, but otherwise, these things are different.

For each j, there is a unique nonzero map in $\mathcal{U}$, $f_j : \widetilde{H}^{\mathbb{R}}P^\infty \to J(2^j)$. The map $\mathrm{Sq}^{2^j} : H^{2^j}\mathbb{R}P^\infty \to H^{2^{j+1}}\mathbb{R}P^\infty$ makes the diagram

$$\begin{array}{ccc} \widetilde{H}^*\mathbb{R}P^\infty & \xrightarrow{f_{j+1}} & J(2^{j+1}) \\ & \underset{f_j}{\searrow} & \downarrow{\scriptstyle \cdot\,\mathrm{Sq}^{2^j}} \\ & & J(2^j) \end{array}$$

commute. So we get a map $\widetilde{H}^*\mathbb{R}P^\infty \to K(1)$.

Proposition 3. *$K(n)$ is injective.*

Proof. For $M \in \mathcal{U}$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}(M, K(n)) &\cong \mathrm{Hom}_{\mathcal{U}}(M, \lim J(2^j n)) \\ &\cong \lim \mathrm{Hom}_{\mathcal{U}}(M, J(2^j n)) \\ &\cong \lim (M^{2^j n})^* \\ &\cong (\operatorname{colim} M^{2^j n})^*, \end{aligned}$$

which is an exact functor. □

Theorem 4 (Carlsson)**.** *$\widetilde{H}^*\mathbb{R}P^\infty \to K(1)$ is split.*

Before we prove this, we need a construction and some notation. The map $\cdot \mathrm{Sq}^n : J(2n) \to J(n)$ is given as follows. $J(n) \subseteq J(\bullet)^* \cong \mathbb{F}_2[\xi_0, \xi_1, \dots]$ is spanned by the monomials of weight n, where the **weight** of $\xi_0^{i_0} \cdots \xi_n^{i_n}$ is

$$\mathrm{wt}(\xi_0^{i_0} \cdots \xi_n^{i_n}) = i_0 + 2i_1 + 2^2 i_2 + \cdots + 2^n i_n.$$

We have

$$\cdot \mathrm{Sq}^n(\xi_0^{i_0} \cdots \xi_n^{i_n}) = \begin{cases} 0 & i_0 \neq 0 \\ \xi_0^{i_1} \xi_1^{i_2} \cdots \xi_{n_1}^{i_n} & i_0 = 0. \end{cases}$$

For example,

$$\begin{array}{ccc}
\xi_0^4 & & \\
& & \\
\xi_1 \xi_0^2 & & \\
& & \\
\xi_1^2 & \longrightarrow & \xi_0^2 \\
& & \\
\xi_2 & \longrightarrow & \xi_1^2
\end{array}$$

$$\begin{array}{ccccc}
\Sigma J(3) & \longrightarrow & J(4) & \longrightarrow & J(2) \\
\| & & & & \\
\Sigma^2 J(2) & & & &
\end{array}$$

Thus, $\cdot \mathrm{Sq}^n$ is the restriction of a ring map

$$V : \mathbb{F}_2[\xi_0, \xi_1, \dots] \to \mathbb{F}_2[\xi_0, \xi_1, \dots]$$

with $V(\xi_i) = 0$ for $i = 0$ and ξ_{i-1} otherwise.

This map has the defect that it doesn't preserve one of the gradings (recall $|\xi_i| = (2^i, 1)$). In fact, V halves the first degree, which is a fancy way of saying that it goes from $J(2n)$ to $J(n)$.

Let's fix this! Define

$$J_{1/2^n}(\bullet)^* = \mathbb{F}_2[\xi_{-n}, \xi_{-n+1}, \dots, \xi_0, \xi_1, \dots]$$

where $|\xi_i| = (2^i, 1)$. This is bigraded over $\mathbb{N}\left[\frac{1}{2}\right] \times \mathbb{N}$. Define $J_{1/2^n}(\bullet)^* \to J_{1/2^{n-1}}(\bullet)^*$ to be the ring map given by $\xi_i \mapsto \xi_i$. In particular, ξ_{-n} goes to 0. Then we can define

$$K(\bullet)^* = \lim_n J_{1/2^n}(\bullet)^* \cong \mathbb{F}_2[\dots, \xi_{-1}, \xi_0, \xi_1, \dots].$$

This has a vertical unstable module structure with $\mathrm{Sq}^1 \xi_i = \xi_{i-1}^2$.

For example,

$$K(1)^* = \lim\{\cdots \to J_{1/4}(1) \to J_{1/2}(1) \to J_1(1)\},$$

but each $J_{1/2^n}(1)$ is isomorphic to $J(2^n)$ after changing the gradings, so we get our ordinary $K(1)^*$.

Example 5. In degree $(1, *)$ in $K(\bullet)^*$, we have the following elements:

$$\vdots$$

$$\xi_{-2}^4$$

$$\xi_{-2}^2\xi_{-1}$$

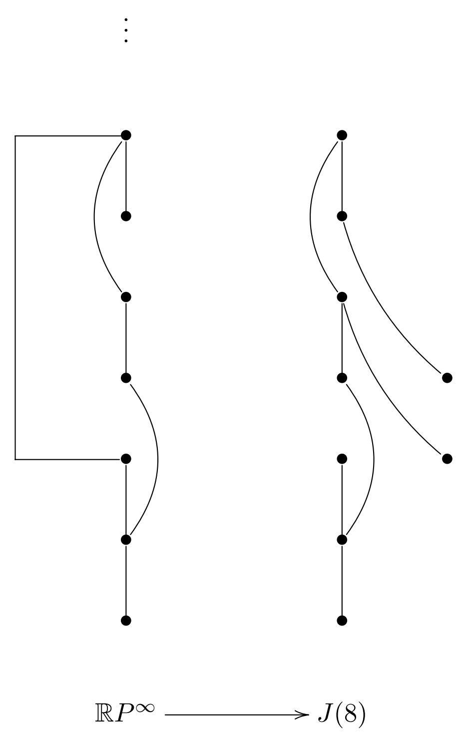
$$\xi_{-1}^2$$

$$\xi_0$$

Proof of Carlsson's theorem. Define $g : K(\bullet)^* \to \widetilde{H}^*(\mathbb{R}P^\infty) = \mathbb{F}_2[x]$ to be the unique algebra map with $g(\xi_i) = x$ for all i. This is a map of modules over the Steenrod algebra because $g(\mathrm{Sq}^1 \xi_i) = g(\xi_{i-1}^2) = x^2 = \mathrm{Sq}^1(x) = \mathrm{Sq}^1 g(\xi_i)$. So it's a map in $\mathcal{U}$. We also have a map

$$f : \widetilde{H}^*(\mathbb{R}P^\infty) \to K(1) \to K(\bullet)^*.$$

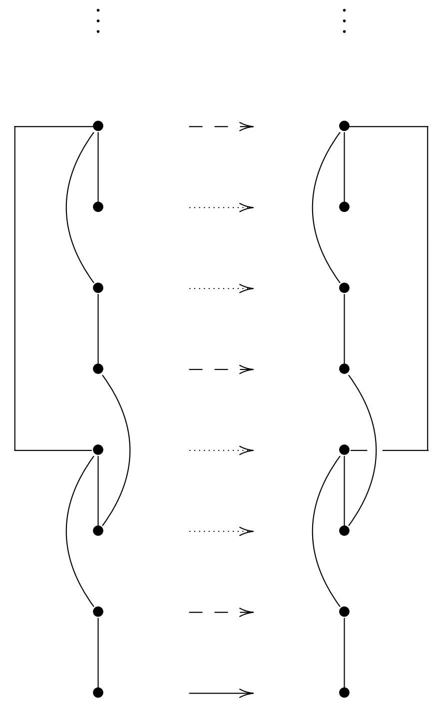
I claim that $H^1\mathbb{R}P^\infty \xrightarrow{f} K(1)^1 \xrightarrow{g} H^1\mathbb{R}P^\infty$ is the identity. We can see this through a diagram.



In the diagram, $x \mapsto \xi_3 \in J(8)$ (the lowest cells), which becomes ξ_0 in $K(1)^*$. We always have $x \mapsto \xi_0 \mapsto x$. The theorem then follows from

Lemma 6. *Any map in $\mathcal{U}$ from $\widetilde{H}^*\mathbb{R}P^\infty \to \widetilde{H}^*\mathbb{R}P^\infty$ which is nonzero in degree 1 is an isomorphism.*

Rather than proving this, let's just draw a picture.



The map is an isomorphism in degree 1, and a map of Steenrod modules, hence the dashed arrows are also isomorphisms by applying Steenrod operations; the dotted arrows are isomorphisms as well.

More formally, if $f(x) = x$, then

$$f(x^{2^s}) = f(\mathrm{Sq}^{2^{s-1}}\,\mathrm{Sq}^{2^{s-2}}\cdots\mathrm{Sq}^1(x)) = \mathrm{Sq}^{2^{s-1}}\,\mathrm{Sq}^{2^{s-2}}\cdots\mathrm{Sq}^1(f(x)) = x^{2^s}.$$

Then for $1 \leq j \leq 2^{s-1}$, write $j = 2^{t_1} + \cdots + 2^{t_i}$, with $t_1 > \cdots > t_i$. Using that $\mathrm{Sq}^{2^{t_1}}\cdots\mathrm{Sq}^{2^{t_i}}\,x^{2^s-j} = x^{2^s}$, we get that $f(x^{2^s-j}) = x^{2^s-j}$ as well. This completes the proof. □

Remark 7. The point is that the Steenrod module diagram for $\widetilde{H}^*\mathbb{R}P^\infty$ is connected. In characteristic $p > 0$, this is no longer true – in fact, we'll have $p-1$ summands, and we have to check that the bottom elements of all of them are hit. Otherwise, the ideas of the proof are the same.

One version of the Sullivan conjecture is as follows. Let X be a finite pointed 1-connected CW-complex; then for all $t \geq 0$, $[\Sigma^t\mathbb{R}P^\infty, X]_* = *$ (that is, pointed maps). The first algebraic approximation is the Hurewicz map:

$$[\Sigma^t\mathbb{R}P^\infty, X]_* \to \mathrm{Hom}_{\mathcal{U}}(\widetilde{H}^*X, \Sigma^t\widetilde{H}^*\mathbb{R}P^\infty) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega^t\widetilde{H}^*X, \widetilde{H}^*\mathbb{R}P^\infty).$$

Lemma 8. *If M is finite, so is $\Omega^t_n M$, the n-fold derived t-fold loops functor of M.*

Assuming this, for $x \in H^nX$, then for some large s, $\mathrm{Sq}^{2^s n}\cdots\mathrm{Sq}^n\,x = 0$, by finiteness. If $f : \Omega^t\widetilde{H}^*X \to \widetilde{H}^*\mathbb{R}P^\infty$ is any map, then $\mathrm{Sq}^{2^s n}\cdots\mathrm{Sq}^n\,f(x) = f(\mathrm{Sq}^{2^s n}\cdots\mathrm{Sq}^n\,x) = 0$, but the squares act injectively on $\widetilde{H}^*\mathbb{R}P^\infty$, so $f(x) = 0$ as well. Thus, any map $\Sigma^t\mathbb{R}P^\infty \to X$ is at least zero on cohomology.

We've seen that $\widetilde{H}^*\mathbb{R}P^\infty$ is injective, but $\Sigma^t\widetilde{H}^*\mathbb{R}P^\infty$ is not. Nevertheless, next time we'll prove the following.

Proposition 9. *If M is finite, then $\mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma^t\widetilde{H}^*\mathbb{R}P^\infty) = 0$ for all $s, t \geq 0$.*