Lecture 7: Injectives and adjunctions

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One version of the Sullivan conjecture, which we'll greatly generalize, is that if X is a finite CW-complex, then the space of pointed maps $\mathrm{map}_*(B\mathbb{Z}/p, X)$ is contractible. That is,

$$[\Sigma^t B\mathbb{Z}/p, X]_* = 0$$

for all t.

As an algebraic warm-up, we can treat this in cohomology instead of homotopy.

Definition 1. $M \in \mathcal{U}$ is **locally finite** if for all $x \in M$, the submodule $\mathcal{A}x \subseteq M$ is a finite set.

(This includes things like infinite sums and products of finite modules.)

Proposition 2. *If M is locally finite and finite type, then $\mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma^t \widetilde{H}^* B\mathbb{Z}/p) = 0$ for all $s \geq 0$ and all t.*

Before proving this, let's have a little fun. Let $\Omega^s = \Omega \circ \cdots \circ \Omega$ be the left adjoint to Σ^s (acting on $\mathcal{U}$, so the 'left' and 'right' are switched from how they are in topology). Recall that Ω^s_t is the tth left derived functor of Ω^s.

Proposition 3. *For all $s \geq 1$, there is a short exact sequence*

$$0 \to \Omega(\Omega^s_t M) \to \Omega^{s+1}_t M \to \Omega_1 \Omega^s_{t-1} M \to 0.$$

(This is a degenerate case of a composition of derived functors spectral sequence.)

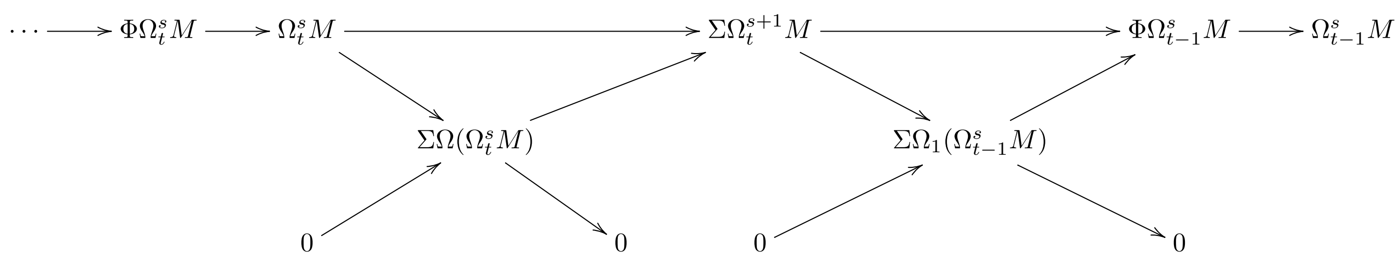
Proof. (Take $p = 2$.) Recall that we have

$$0 \to \Sigma\Omega_1 M \to \Phi M \xrightarrow{\lambda} M \to \Sigma\Omega M \to 0.$$

Let $P_\bullet \to M$ be a projective resolution; we get

$$0 \to \Phi\Omega^s P_\bullet \to \Omega^s P_\bullet \to \Sigma\Omega^{s+1} P_\bullet \to 0$$

(the first term vanishes since $\Omega^s P_\bullet$ is projective [since $\Omega F(k+1) = F(k)$], and Ω_1 vanishes on projectives). As Φ is an exact functor, taking the homology of this exact sequence of complexes gives us



and this splits into short exact sequences as shown, proving the claim. ∎

Corollary 4. *1. $\Omega^s_t M = 0$ if $t > s$, and*

$$\Omega^s_s M = \underbrace{\Omega_1 \cdots \Omega_1}_{s} M.$$

2. If M is locally finite, so is $\Omega^s_t M$.

Proof. 1. follows from induction on s, since we know that it's true for $s = 1$.

2. If M is locally finite, then so is $\Phi(M)$, and hence so is $\Omega_1 M$, and we can use induction. □

Proof of Proposition 2. Let $P_\bullet \to M$ be a projective resolution. Then

$$\mathrm{Hom}_{\mathcal{U}}(P_\bullet, \Sigma^t \widetilde{H}^* \mathbb{R}P^\infty) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega^t P_\bullet, \widetilde{H}^* \mathbb{R}P^\infty).$$

Taking cohomology gives

$$H^s \mathrm{Hom}_{\mathcal{U}}(P_\bullet, \Sigma^t \widetilde{H}^* \mathbb{R}P^\infty) \cong H^s Hom_{\mathcal{U}}(\Omega^t P_\bullet, \widetilde{H}^* \mathbb{R}P^\infty) \cong \mathrm{Hom}_{\mathcal{U}}(H_s \Omega^t P_\bullet, \widetilde{H}^* \mathbb{R}P^\infty),$$

the last isomorphism since $\widetilde{H}^* \mathbb{R}P^\infty$ is injective. This, in turn, is isomorphic to $\mathrm{Hom}_{\mathcal{U}}(\Omega^t_s M, \widetilde{H}^* \mathbb{R}P^\infty)$. Finally, $\Omega^t_s M$ is locally finite by Corollary 4, so it has no nonzero maps to the injective module $\widetilde{H}^* \mathbb{R}P^\infty$ – any $x \in \widetilde{H}^* \mathbb{R}P^\infty$ has $\mathcal{A}x \cong \mathcal{A}$. □

We can generalize the statement that $\widetilde{H}^* B\mathbb{Z}/p$ is injective to the following: for all finite $\mathbb{F}_p$-vector spaces V and all $n \geq 0$, $J(n) \otimes H^* BV$ is injective.

Definition 5. A module $M \in \mathcal{U}$ is **reduced** if $\lambda : \Phi M \to M$ is an injection. (Equivalently, if for $x \in M^n$, $\mathrm{Sq}^n x \neq 0$).

For example, $\widetilde{H}^* \mathbb{R}P^\infty$ and $F(k)$ are reduced.

Theorem 6 (Lannes-Zarati)**.** *If K is a reduced injective and J is injective, and one of them is finite type, then $K \otimes J$ is injective.*

The Lannes-Zarati theorem implies that $J(n) \otimes H^* BV$ is injective for a finite-dimensional vector space V. Indeed, $J(0) = \mathbb{F}_p$ is a reduced injective, so $H^* B\mathbb{F}_p = J(0) \oplus \widetilde{H}^* B\mathbb{F}_p$ is a reduced injective, so $H^* B\mathbb{F}_p^n \cong (H^* B\mathbb{F}_p)^{\otimes n}$ is injective, by Lannes-Zarati, and one easily sees that it's reduced. By Lannes-Zarati again, $J(n) \otimes H^* B\mathbb{F}_p^n$ is injective.

Fun with adjoints

We know that if $F : \mathcal{U}^{\mathrm{op}} \to \mathbb{F}_p-\mathsf{VectorSpaces}$ is a functor that sends sums to products and surjections to injections, then F is representable, i. e. there's a $J_F \in \mathcal{U}$ and an isomorphism $\mathrm{Hom}_{\mathcal{U}}(M, J_F) \cong F(M)$. (Remember that this is how we got the Brown-Gitler modules. You can explicitly construct J_F by $J^k_F = \mathrm{Hom}_{\mathcal{U}}(F(k), J_F) = F(F(k))$, and the Steenrod operations are induced by the relevant maps between the $F(k)$.)

Corollary 7. *If $\Psi : \mathcal{U} \to \mathcal{U}$ preserves surjections (i. e. is right exact) and sends sums to sums, then Ψ has a right adjoint $\widetilde{\Psi}$.*

Proof. Define $F : \mathcal{U}^{\mathrm{op}} \to \mathbb{F}_p-\mathsf{VectorSpaces}$ by $F(M) = \mathrm{Hom}_{\mathcal{U}}(\Psi(M), N)$. This satisfies the conditions above, so we get $\mathrm{Hom}_{\mathcal{U}}(\Psi(M), N) \cong \mathrm{Hom}_{\mathcal{U}}(M, \widetilde{\Psi}(N))$. We just have to show that $\widetilde{\Psi}$ is a functor, but this follows from the Yoneda lemma. (If you've read Adams's blue book about spectra, this is how he constructs Spanier-Whitehead duality.) □

Example 8. As a result, Φ has a right adjoint $\widetilde{\Phi}$. Surprisingly, Σ has a right adjoint $\widetilde{\Sigma}$ as well as its left adjoint Ω. Let's look at how these behave.

First, we have

$$\mathrm{Hom}_{\mathcal{U}}(M, \widetilde{\Phi}(J(2n))) \cong \mathrm{Hom}_{\mathcal{U}}(\Phi(M), J(2n)) \cong (\Phi(M)^{2n})^* \cong (M^n)^* \cong \mathrm{Hom}_{\mathcal{U}}(M, J(n)).$$

So $\widetilde{\Phi}(J(2n)) = J(n)$ (and likewise, $\widetilde{\Phi}(J(2n+1)) = 0$).

By the same argument,

$$\mathrm{Hom}_{\mathcal{U}}(M, \widetilde{\Sigma}(J(n))) \cong \mathrm{Hom}_{\mathcal{U}}(\Sigma M, J(n)) \cong ((\Sigma M)^n)^* \cong (M^{n-1})^* \cong \mathrm{Hom}_{\mathcal{U}}(M, J(n-1)).$$

So $\widetilde{\Sigma}(J(n)) \cong J(n-1)$.

The adjoint of $1 : \widetilde{\Sigma}M \to \widetilde{\Sigma}M$ is a map $\Sigma\widetilde{\Sigma}M \to M$. The adjoint of $\lambda : \Phi M \to M$ is a map $\widetilde{\lambda} : M \to \widetilde{\Phi}M$.

Proposition 9. *The sequence* $0 \to \Sigma\widetilde{\Sigma}M \to M \xrightarrow{\widetilde{\lambda}} \widetilde{\Phi}M$ *is exact.*

(You should check this.

$\Sigma\widetilde{\Sigma}M$ is the largest submodule of M which is a suspension.

Corollary 10. *If* $M \in \mathcal{U}$*, the following are equivalent:*

1. *M is reduced;*
2. *$\lambda_M : \Phi M \to M$ is injective;*
3. *$\widetilde{\lambda}_M : M \to \widetilde{\Phi}M$ is injective;*
4. *$\widetilde{\Sigma}M = 0$.*

Proof. $1 \Rightarrow 4$ because the top Steenrod operation vanishes on any suspension. $4 \Rightarrow 3$ by the exact sequence above. The remaining steps will be left to the next class. We'll then use this to show that $K(1)$ is injective. $\square$