Lecture 8: The tensor product of injective modules

October 15, 2014

We're trying to prove the following.

Theorem 1. *If K is a reduced injective and J is injective, and both are finite type, then $K \otimes J$ is injective.*

This will show, for example, that $H^*BV \otimes J(n)$.

Last time, we introduced right adjoints $\widetilde{\Sigma}$, $\widetilde{\Phi}$ to suspension Σ and doubling Φ. $\Sigma\widetilde{\Sigma}M \subseteq M$ is the largest submodule that is a suspension, and we also had

$$\widetilde{\Phi}J(2n) = J(n), \qquad \widetilde{\Phi}J(2n+1) = 0, \qquad \widetilde{\Sigma}J(n) = J(n-1).$$

Lemma 2. *The sequence $0 \to \Sigma\widetilde{\Sigma}M \to M \xrightarrow{\widetilde{\lambda}} \widetilde{\Phi}M$ is exact.*

Proof. First suppose that $M = J(2n)$. Then this sequence is

$$0 \to \Sigma J(2n-1) \to J(2n) \to J(n) \to 0,$$

which is actually exact on the right as well. If $M = J(2n+1)$, then we instead get

$$0 \to \Sigma J(2n) \xrightarrow{\sim} J(2n+1) \to 0 \to 0,$$

which is also short exact. Since every injective is a split summand of a product of the $J(n)$, the sequence is short exact for any injective M. For general M, take an injective resolution $M \to I^\bullet$. Applying the functors gives an exact sequence

$$0 \to \Sigma\widetilde{\Sigma}I^\bullet \to I^\bullet \to \widetilde{\Phi}I^\bullet \to 0.$$

Σ is exact and both $\widetilde{\Sigma}$ and $\widetilde{\Phi}$ are left exact, so the result follows, as the beginning of the long exact sequence on cohomology of this short exact sequence of complexes. In fact, we can finish the sequence:

$$0 \to \Sigma\widetilde{\Sigma}M \to M \to \widetilde{\Phi}M \to R^1(\Sigma\widetilde{\Sigma})M \to 0.$$

□

Proposition 3. *The following are equivalent:*

1. *M is reduced.*
2. *$\lambda : \Phi M \to M$ is injective (equivalently, $\Omega_1 M = 0$).*
3. *$\widetilde{\lambda} : M \to \widetilde{\Phi}M$ is injective.*
4. *$\widetilde{\Sigma}M = 0$.*

Proof. $1 \Leftrightarrow 4$ since a reduced module has no submodule which is a suspension. $4 \Leftrightarrow 3$ is the above lemma. $1 \Rightarrow 2$ is the definition of reduced, since λ_M sends things to their top Steenrod power. The rest is an exercise. □

Corollary 4. *The Carlsson module $K(n) = \lim J(2^i n)$ is reduced.*

Proof. Indeed, $\widetilde{\Phi}$ is a right adjoint, so commutes with limits. We get

$$K(n) \to \widetilde{\Phi}K(n) = \lim \widetilde{\Phi}J(2^i n) = \lim J(2^{i-1}n) = K(n),$$

which can be checked to be the identity. ☐

Exercise 5. Every reduced module is a submodule of a module $\prod_\alpha K(n_\alpha)$.

'Proof' of Theorem 1. We're assuming that K and J are injective and finite type, with K reduced, and we want to show that $K \otimes J$ is injective.

Step 1. If K is finite type and reduced, then K is a split submodule of some $\bigoplus_\alpha K(n_\alpha)$, and J is a split submodule of some $\bigoplus_\alpha J(m_\alpha)$. So we reduce to the case $K(n) \otimes J(k)$.

Step 2. Define

$$L(n,k) = \lim\{\cdots \to J(2^{i+1}n + k) \xrightarrow{\cdot \mathrm{Sq}^{2^i n}} J(2^i n + k) \to \cdots\}.$$

This is injective, since $K(n) = L(n, 0)$ is injective.

Step 3. Consider the diagram

$$\begin{tikzcd}
J(2^{i+1}n) \otimes J(k) \arrow[r] \arrow[d, "\cdot \mathrm{Sq}^{2^i n} \otimes 1"'] & J(2^{i+1}n + k) \arrow[d, "\cdot \mathrm{Sq}^{2^i n}"] \\
J(2^i n) \otimes J(k) \arrow[r] & J(2^i n + k).
\end{tikzcd}$$

You'd like to say this commutes, but it doesn't. Instead, you can show that there is an integer $q(n,k)$ such that this commutes in degree $\leq i + q(n,k)$. Hence we get a map $K(n) \otimes J(k) \to L(n,k)$.

Step 4. $K(n) \otimes J(k) \to L(n,k)$ is an isomorphism. This is certainly true if $k = 0$, and we use induction on k to get it in general, using

$$0 \to \Sigma J(2n-1) \to J(2n) \to J(n) \to 0.$$

(By the way, this is called the **Mahowald exact sequence**, though he actually worked with its Spanier-Whitehead dual.) ☐

A turn to topology

In some appropriate category of spaces, the functor $Y \mapsto X \times Y$ has a right adjoint, so that continuous maps $X \times Y \to Z$ correspond to continuous maps $X \to \mathrm{map}(Y, Z)$. (This doesn't work in the category of all spaces, but it does work in simplicial sets, or in compactly generated weak Hausdorff spaces with the compactly generated product.) In fact, there's a homeomorphism

$$\mathrm{map}(X \times Y, Z) \cong \mathrm{map}(X, \mathrm{map}(Y, Z)).$$

(These are unpointed maps, not pointed maps!)

If X and Y are spaces of finite type, then we also have

$$H^*(X \times Y) \cong H^*X \otimes H^*Y$$

in the category $\mathcal{K}$ of unstable algebras. One can likewise ask whether the functor $L \mapsto K \otimes L$ on $\mathcal{K}$ has a left adjoint.

In fact, it does, but we'll first answer an easier question: does $N \mapsto M \otimes N$ on $\mathcal{U}$ have a left adjoint?

Theorem 6. *For M of finite type, there is a functor $L \mapsto (M, L)_{\mathcal{U}}$ so that*

$$\mathrm{Hom}_{\mathcal{U}}((M, L)_{\mathcal{U}}, N) \cong \mathrm{Hom}_{\mathcal{U}}(L, M \otimes N).$$

Proof. The functor $N \mapsto M \otimes N$ is exact and preserves products. ☐

Definition 7. **Lannes' T-functor** is defined, for a finite $\mathbb{F}_p$-vector space V, as

$$T_V M = (H^* BV, M)_{\mathcal{U}}.$$

In particular, $\mathrm{Hom}_{\mathcal{U}}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^* BV \otimes N)$.

Theorem 8 (Lannes)**.** *T_V is exact and preserves colimits and tensor products.*

Exact isn't hard to prove at this point, and all left adjoints preserve colimits, but tensor products takes some work.

Corollary 9. $\mathrm{Hom}_{\mathcal{K}}(T_V K, L) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^* BV \otimes L)$ *– so T_V naturally acts on the category of unstable algebras, and is a left adjoint there as well.*

Remark 10. Let X be a space of finite type. Then there is an evaluation map

$$\mathrm{map}(BV, X) \times BV \to X,$$

which induces

$$H^* X \to H^* \mathrm{map}(BV, X) \otimes H^* BV.$$

Adjunction gives

$$T_V H^* X \to H^* \mathrm{map}(BV, X).$$

This is an isomorphism surprisingly often.

Theorem 11 (Miller)**.** *If X is a finite CW-complex, then $T_V H^* X \cong H^* X$, and the map*

$$H^* X \cong T_V H^* X \to H^* \mathrm{map}(BV, X)$$

is an isomorphism.

Corollary 12. *If X is 1-connected, then the inclusion of the constant maps $X \to \mathrm{map}(BV, X)$ is a weak equivalence.*

This is a great example of the Bill Dwyer approach to homotopy: to find homotopy classes of maps between two spaces, calculate the whole homotopy type of the mapping space, and then read off the π_0.

Corollary 13. *If X is a pointed space, then there is a fibration*

$$\mathrm{map}_*(BV, X) \to \mathrm{map}(BV, X) \to X$$

where the last map is evaluation on the basepoint. Thus, if X is a 1-connected finite pointed CW-complex, $\mathrm{map}_*(BV, X) \simeq *$.