Lecture 9: Properties of the T-functor

October 17, 2014

This is all coming out of Lionel Schwartz's book, by the way.

Last time we were talking about Lannes's T-functor. If V is a finite-dimensional $\mathbb{F}_p$-vector space, then $T_V : \mathcal{U} \to \mathcal{U}$ is characterized by

$$\mathrm{Hom}_{\mathcal{U}}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^* BV \otimes N).$$

If you Google Lannes, you unfortunately won't find him.

Figure 1: A more famous Lannes.

Proposition 1. *T_V is exact and commutes with colimits.*

Proof. Every left adjoint commutes with colimits.

For exactness, suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, and look at the sequence

$$(*) \qquad \begin{array}{ccccc} \mathrm{Hom}_{\mathcal{U}}(T_V M_1, J(n)) & \longleftarrow & \mathrm{Hom}_{\mathcal{U}}(T_V M_2, J(n)) & \longleftarrow & \mathrm{Hom}_{\mathcal{U}}(T_V M_3, J(n)) \\ \| & & \| & & \| \\ \mathrm{Hom}_{\mathcal{U}}(M_1, H^* BV \otimes J(n)) & \longleftarrow & \mathrm{Hom}_{\mathcal{U}}(M_2, H^* BV \otimes J(n)) & \longleftarrow & \mathrm{Hom}_{\mathcal{U}}(M_3, H^* BV \otimes J(n)) \end{array}$$

Because $H^* BV \otimes J(n)$ is exact, (*) is exact. $\square$

Example/Theorem 2. *If M is finite, then $T_V M \cong M$.*

(This is an algebraic analog of the fact that $\mathrm{map}(BV, X) \simeq X$ for X a finite CW-complex.)

Proof. We can reduce to the case where $M = \Sigma^d \mathbb{F}_p$, by using exactness of T_V and the short exact sequences

$$0 \to M^n \to M^{\leq n} \to M^{\leq (n-1)} \to 0,$$

and noting that $M^n = \Sigma^n W$ for a finite-dimensional vector space W. (This is the dual of the skeleton filtration of a CW-complex.) Now,

$$\mathrm{Hom}_{\mathcal{U}}(T_V \Sigma^d \mathbb{F}_p, N) \cong \mathrm{Hom}_{\mathcal{U}}(\Sigma^d \mathbb{F}_p, H^* BV \otimes N) \cong \mathrm{Hom}_{\mathcal{U}}(\Sigma^d \mathbb{F}_p, N) \oplus \mathrm{Hom}_{\mathcal{U}}(\Sigma^d \mathbb{F}_p, \widetilde{H}^* BV \otimes N).$$

The second term is zero, since every element $x \in \widetilde{H}^* BV \otimes N$ has infinitely many Steenrod operations θ with $\theta(x) \neq 0$, while all the Steenrod operations above $\mathcal{P}^0$ vanish on $\Sigma^d \mathbb{F}_p$. Thus,

$$\mathrm{Hom}_{\mathcal{U}}(T_V \Sigma^d \mathbb{F}_p, N) \cong \mathrm{Hom}_{\mathcal{U}}(\Sigma^d \mathbb{F}_p, N).$$

By the Yoneda lemma, $T_V \Sigma^d \mathbb{F}_p \cong \Sigma^d \mathbb{F}_p$. □

(For example, $\widetilde{H}^* \mathbb{R}P^\infty = u\mathbb{F}_2[u]$, and any element of $\widetilde{H}^* \mathbb{R}P^\infty \otimes N$ is of the form $x = \sum_{i=1}^n u^i \otimes x_i$, and then $\mathrm{Sq}^{2^k n} \cdots \mathrm{Sq}^n(x) \neq 0$ for all $k \geq 0$.)

Theorem 3 (Lannes)**.** *T_V commutes with tensor products.*

Corollary 4. *If K is an unstable algebra, then so is $T_V K$.*

(If this is obvious, it shouldn't be – you shouldn't even believe it, really. It's something special about V.) The multiplication on $T_V K$ is given by

$$T_V K \otimes T_V K \overset{\sim}{\leftarrow} T_V(K \otimes K) \overset{T_V m}{\to} T_V K,$$

where m is the multiplication on K.

Theorem 5. *The resulting functor $T_V : \mathcal{K} \to \mathcal{K}$ (on unstable algebras, now) is left adjoint to*

$$L \mapsto H^* BV \otimes L,$$

so that there's a natural isomorphism of sets

$$\mathrm{Hom}_{\mathcal{K}}(T_V K, L) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^* BV \otimes L).$$

Before we prove these theorems, let's look at some properties of T_V. First, T_V commutes with suspension and loops, because these are both left adjoints, and $H^* BV \otimes \cdot$ commutes with their right adjoints $\widetilde{\Sigma}$ and Σ.

Proposition 6. *$T_V \Phi M \cong \Phi T_V M$.*

This takes some work. Let $p = 2$. Define $Q_i \in \mathcal{A}$ by $Q_0 = \mathrm{Sq}^1$, and

$$Q_i = \mathrm{Sq}^{2^i} Q_{i-1} + Q_{i-1} \mathrm{Sq}^{2^i}.$$

For example, $Q_1 = \mathrm{Sq}^1 \mathrm{Sq}^2 + \mathrm{Sq}^2 \mathrm{Sq}^1 = \mathrm{Sq}^3 + \mathrm{Sq}^2 \mathrm{Sq}^1$.

These things have some nice properties.

1. $Q_i^2 = 0$ and $Q_i Q_j = Q_j Q_i$.
2. If K is an unstable algebra, then on K, $Q_i(xy) = Q_i x \cdot y + x \cdot Q_i y$, i. e., the Q_i are derivations.
3. The left ideal in $\mathcal{A}$ generated by the Q_i is the two-sided ideal generated by Sq^1. (For an example of this,
$$\mathrm{Sq}^1 \mathrm{Sq}^4 = \mathrm{Sq}^5 = \mathrm{Sq}^2 \mathrm{Sq}^3 + \mathrm{Sq}^4 \mathrm{Sq}^1 = \mathrm{Sq}^2 \mathrm{Sq}^3 + \mathrm{Sq}^2 \mathrm{Sq}^2 \mathrm{Sq}^1 + \mathrm{Sq}^4 \mathrm{Sq}^1$$
[since $\mathrm{Sq}^2 \mathrm{Sq}^2 \mathrm{Sq}^1 = \mathrm{Sq}^3 \mathrm{Sq}^1 \mathrm{Sq}^1 = 0$], and this is equal to $\mathrm{Sq}^2 Q_1 + \mathrm{Sq}^4 Q_0$.)

3bis. There is an exact sequence of $\mathcal{A}$-modules

$$0 \to \mathcal{A}\{Q_i\} \to \mathcal{A} \to \Phi\mathcal{A} \to 0.$$

In fact, it's even easier to see that the kernel of the right-hand map is the two-sided ideal generated by Sq^1.

(By the way, $\Phi\mathcal{A} = H^*BP$!)

Let $M \in \mathcal{U}$ and

$$F(M)^k = \begin{cases} 0 & k = 2n+1 \\ \{x \in M^{2n} : Q_i x = 0 \text{ for all } i\} & k = 2n. \end{cases}$$

Then $F(M) \subseteq M$ is a sub-unstable $\mathcal{A}$-module. This follows from property 3: indeed, if $\theta \in \mathcal{A}$, then $Q_i\theta \in \mathcal{A}\,\mathrm{Sq}^1\,\mathcal{A}$, so we can write $Q_i\theta = \sum \theta_j Q_j$. Thus, if $Q_j x = 0$ for all j, then $Q_i\theta x = 0$ for all i and all θ.

Recall that $\mathrm{Hom}_{\mathcal{U}}(\Phi M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, \widetilde{\Phi} N)$.

Proposition 7. *$\Phi\widetilde{\Phi} M \cong F(M)$.*

Proof. The natural map $\Phi\widetilde{\Phi} M \to M$ factors through $F(M)$, since $\Phi\widetilde{\Phi} M$ is concentrated in even degrees, so all the Q_i, which have odd degrees, vanish on it. Both functors are left exact and preserve products, so it suffices to prove that $\Phi\widetilde{\Phi} J(n) \to F(J(n))$ is an isomorphism. In $J(\bullet)^*$, we have

$$Q_{i+1}\xi_j = \begin{cases} \xi_{j-i}^{2^i} & i \le j \\ 0 & i > j. \end{cases}$$

The calculation follows. □

Corollary 8. *$\widetilde{\Phi}(M \otimes N) \cong \widetilde{\Phi}(M) \otimes \widetilde{\Phi}(N)$.*

The proof uses that Q_i is a derivation.

Corollary 9. *$\widetilde{\Phi}(H^*BV \otimes N) \cong H^*BV \otimes \widetilde{\Phi}(N)$.*

Proof. One calculates that $\widetilde{\Phi} H^*BV = H^*BV$, using Proposition 7, and then uses the previous corollary. □

Proposition 10. *$\Phi T_V M = T_V \Phi M$.*

Proof. This is what we've been trying to prove this whole time. We just saw that the right adjoints to these functors commute. □